Exercise 11

Show that there is no solution of

$$\Delta u = f \quad \text{in } D, \qquad \frac{\partial u}{\partial n} = g \quad \text{on bdy } D$$

in three dimensions, unless

$$\iiint_D f\,dx\,dy\,dz = \iint_{\text{bdy}(D)} g\,dS.$$

(*Hint:* Integrate the equation.) Also show the analogue in one and two dimensions.

Solution

In Three Dimensions

Integrate both sides of the PDE over the volume of D.

$$\iiint_D \Delta u\,dV = \iiint_D f\,dV$$

Rewrite the Laplacian operator: $\Delta = \nabla^2 = \nabla \cdot \nabla$.

$$\iiint_D \nabla \cdot \nabla u\,dV = \iiint_D f\,dV$$

Apply the divergence theorem to the integral on the left side. As a result, the volume integral becomes a closed surface integral over the domain's boundary.

$$\oiint_{\text{bdy } D} \nabla u \cdot \hat{\mathbf{n}}\,dS = \iiint_D f\,dV$$

The integrand on the left side is also known as the normal derivative, that is, $\nabla u \cdot \hat{\mathbf{n}} = \partial u/\partial n$.

$$\oiint_{\text{bdy } D} \frac{\partial u}{\partial n}\,dS = \iiint_D f\,dV$$

The normal derivative is known to be g on the boundary.

$$\oiint_{\text{bdy } D} g\,dS = \iiint_D f\,dV$$

Therefore, for a solution to exist,

$$\iiint_D f\,dx\,dy\,dz = \iint_{\text{bdy}(D)} g\,dS.$$

A physical interpretation for this result is that the rate of thermal energy generated in D must be equal to the rate flowing across the boundary for a steady-state temperature distribution to exist.

In Two Dimensions

Integrate both sides of the PDE over the area of D.

$$\iint_D \Delta u \, dA = \iint_D f \, dA$$

Rewrite the Laplacian operator: $\Delta = \nabla^2 = \nabla \cdot \nabla$.

$$\iint_D \nabla \cdot \nabla u \, dA = \iint_D f \, dA$$

Apply Green's theorem (essentially the divergence theorem in two dimensions) to the integral on the left side. As a result, the area integral becomes a closed loop integral over the domain's boundary.

$$\oint_{\text{bdy } D} \nabla u \cdot \hat{\mathbf{n}} \, ds = \iint_D f \, dA$$

The integrand on the left side is also known as the normal derivative, that is, $\nabla u \cdot \hat{\mathbf{n}} = \partial u / \partial n$.

$$\oint_{\text{bdy } D} \frac{\partial u}{\partial n} \, ds = \iint_D f \, dA$$

The normal derivative is known to be g on the boundary.

$$\oint_{\text{bdy } D} g \, ds = \iint_D f \, dA$$

Therefore, for a solution to exist,

$$\iint_D f \, dx \, dy = \int_{\text{bdy}(D)} g \, ds.$$

A physical interpretation for this result is that the rate of thermal energy generated in D must be equal to the rate flowing across the boundary for a steady-state temperature distribution to exist.

In One Dimension

In one dimension the PDE becomes

$$\frac{d^2u}{dx^2} = f(x), \quad \text{in } a < x < b,$$

and the boundary conditions associated with it are

$$\frac{du}{dx} = A \quad \text{at } x = a$$
$$\frac{du}{dx} = B \quad \text{at } x = b.$$

Integrate both sides of the PDE with respect to x from a to b.

$$\int_a^b \frac{d^2u}{dx^2}\,dx = \int_a^b f(x)\,dx$$

Evaluate the integral on the left side.

$$\left.\frac{du}{dx}\right|_a^b = \int_a^b f(x)\,dx$$

Plug in the limits.

$$\frac{du}{dx}(b) - \frac{du}{dx}(a) = \int_a^b f(x)\,dx$$

du/dx is known at the ends.

$$B - A = \int_a^b f(x)\,dx$$

Therefore, for a solution to exist,

$$\int_a^b f(x)\,dx = B - A.$$

A physical interpretation for this result is that the rate of thermal energy generated in D must be equal to the rate flowing across the ends for a steady-state temperature distribution to exist.