Exercise 4

Solve $u_{xx} + u_{yy} + u_{zz} = 0$ in the spherical shell $0 < a < r < b$ with the boundary conditions $u = A$ on $r = a$ and $u = B$ on $r = b$, where A and B are constants. (*Hint:* Look for a solution depending only on r.)

Solution

The PDE we have to solve is the Laplace equation.

$$\nabla^2 u = 0$$

Since the region we're solving it in is a spherical shell, we will expand the Laplacian operator in spherical coordinates (θ here represents the angle from the polar axis).

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\left[\frac{\partial^2 u}{\partial \theta^2} + (\cot\theta)\frac{\partial u}{\partial \theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2}\right] = 0$$

We assume from the boundary conditions that the solution is spherically symmetric, that is, it only depends on r, $u = u(r)$. Consequently, the PDE simplifies to an ODE that can be solved relatively easily.

$$\frac{d^2u}{dr^2} + \frac{2}{r}\frac{du}{dr} = 0$$

Notice that this is a first-order ODE for du/dr. Multiply both sides by the integrating factor

$$I = \exp\left(\int^r \frac{2}{s}\,ds\right) = \exp(2\ln r) = \exp(\ln r^2) = r^2$$

to get

$$r^2\frac{d^2u}{dr^2} + 2r\frac{du}{dr} = 0.$$

The left side can be written as $d/dr(I du/dr)$ as a result of the product rule.

$$\frac{d}{dr}\left(r^2\frac{du}{dr}\right) = 0$$

Integrate both sides with respect to r.

$$r^2\frac{du}{dr} = C_1$$

Divide both sides by r^2.

$$\frac{du}{dr} = \frac{C_1}{r^2}$$

Integrate both sides with respect to r once more.

$$u(r) = -\frac{C_1}{r} + C_2$$

Apply the boundary conditions here to determine the constants, C_1 and C_2.

$$u(a) = -\frac{C_1}{a} + C_2 = A$$
$$u(b) = -\frac{C_1}{b} + C_2 = B$$

This is a system of two equations for two unknowns. Solving it gives

$$C_1 = \frac{ab(A-B)}{a-b} \quad \text{and} \quad C_2 = \frac{aA-bB}{a-b}.$$

Therefore,

$$u(r) = -\frac{1}{r}\frac{ab(A-B)}{a-b} + \frac{aA-bB}{a-b}.$$