Exercise 8

Solve $u_{xx} + u_{yy} + u_{zz} = 1$ in the spherical shell $a < r < b$ with $u = 0$ on $r = a$ and $\partial u/\partial r = 0$ on $r = b$. Then let $a \to 0$ in your answer and interpret the result.

Solution

The PDE we have to solve is known as the Poisson equation.

$$\nabla^2 u = 1$$

Since the region we're solving it in is a spherical shell, we will expand the Laplacian operator in spherical coordinates (θ here represents the angle from the polar axis).

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\left[\frac{\partial^2 u}{\partial \theta^2} + (\cot\theta)\frac{\partial u}{\partial \theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2}\right] = 1$$

We assume from the boundary conditions that the solution is spherically symmetric, that is, it only depends on r, $u = u(r)$. Consequently, the PDE simplifies to an ODE that can be solved relatively easily.

$$\frac{d^2u}{dr^2} + \frac{2}{r}\frac{du}{dr} = 1$$

Notice that this is a first-order ODE for du/dr. Multiply both sides by the integrating factor

$$I = \exp\left(\int^r \frac{2}{s}\,ds\right) = \exp(2\ln r) = \exp(\ln r^2) = r^2$$

to get

$$r^2\frac{d^2u}{dr^2} + 2r\frac{du}{dr} = r^2.$$

The left side can be written as $d/dr(I du/dr)$ as a result of the product rule.

$$\frac{d}{dr}\left(r^2\frac{du}{dr}\right) = r^2$$

Integrate both sides with respect to r.

$$r^2\frac{du}{dr} = \frac{r^3}{3} + C_1$$

Divide both sides by r^2.

$$\frac{du}{dr} = \frac{r}{3} + \frac{C_1}{r^2}$$

Apply the boundary condition at $r = b$ now to determine C_1.

$$\frac{du}{dr}(b) = \frac{b}{3} + \frac{C_1}{b^2} = 0 \quad \to \quad C_1 = -\frac{b^3}{3}$$

The formula for du/dr becomes

$$\frac{du}{dr} = \frac{r}{3} - \frac{b^3}{3r^2}$$

Integrate both sides with respect to r once more.

$$u(r) = \frac{r^2}{6} + \frac{b^3}{3r} + C_2$$

Apply the boundary condition at $r = a$ to determine C_2.

$$u(a) = \frac{a^2}{6} + \frac{b^3}{3a} + C_2 = 0 \quad \rightarrow \quad C_2 = -\frac{a^2}{6} - \frac{b^3}{3a}$$

So then

$$u(r) = \frac{r^2}{6} + \frac{b^3}{3r} - \frac{a^2}{6} - \frac{b^3}{3a}.$$

Therefore,

$$u(r) = \frac{1}{6}(r^2 - a^2) + \frac{b^3}{3}\left(\frac{1}{r} - \frac{1}{a}\right)$$

or

$$u(x, y, z) = \frac{1}{6}(x^2 + y^2 + z^2 - a^2) + \frac{b^3}{3}\left(\frac{1}{\sqrt{x^2 + y^2 + z^2}} - \frac{1}{a}\right).$$

In the limit as $a \to 0$, $u \to -\infty$. The Poisson equation we solved is the governing equation for the steady-state temperature in a spherical shell that has a constant heat source (technically a heat sink). The boundary condition, $u(a) = 0$, means the temperature is specfied at the inner radius. The boundary condition, $\partial u/\partial r(b) = 0$, means the sphere is insulated at the outer radius, implying that no heat can enter or exit here. If $a \to 0$, then the spherical shell becomes a sphere, and the boundary condition at $r = a$ is effectively lost. Since no heat can enter the sphere at $r = b$, the temperature drops indefinitely due to the heat sink.