Exercise 11

Show that there is no solution of

$$\Delta u = f \quad \text{in } D, \quad \frac{\partial u}{\partial n} = g \quad \text{on } \text{bdy } D$$

in three dimensions, unless

$$\iiint_{D} f \, dx \, dy \, dz = \iint_{\text{bdy}(D)} g \, dS.$$

(*Hint: Integrate the equation.*) Also show the analogue in one and two dimensions.

Solution

**In Three Dimensions**

Integrate both sides of the PDE over the volume of $D$.

$$\iiint_{D} \Delta u \, dV = \iiint_{D} f \, dV$$

Rewrite the Laplacian operator: $\Delta = \nabla^2 = \nabla \cdot \nabla$.

$$\iiint_{D} \nabla \cdot \nabla u \, dV = \iiint_{D} f \, dV$$

Apply the divergence theorem to the integral on the left side. As a result, the volume integral becomes a closed surface integral over the domain’s boundary.

$$\iint_{\text{bdy } D} \nabla u \cdot \hat{n} \, dS = \iiint_{D} f \, dV$$

The integrand on the left side is also known as the normal derivative, that is, $\nabla u \cdot \hat{n} = \frac{\partial u}{\partial n}$.

$$\iint_{\text{bdy } D} \frac{\partial u}{\partial n} \, dS = \iiint_{D} f \, dV$$

The normal derivative is known to be $g$ on the boundary.

$$\iint_{\text{bdy } D} g \, dS = \iiint_{D} f \, dV$$

Therefore, for a solution to exist,

$$\iiint_{D} f \, dx \, dy \, dz = \iint_{\text{bdy}(D)} g \, dS.$$ 

A physical interpretation for this result is that the rate of thermal energy generated in $D$ must be equal to the rate flowing across the boundary for a steady-state temperature distribution to exist.

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In Two Dimensions

Integrate both sides of the PDE over the area of \( D \).

\[
\iint_D \Delta u \, dA = \iint_D f \, dA
\]

Rewrite the Laplacian operator: \( \Delta = \nabla^2 = \nabla \cdot \nabla \).

\[
\iint_D \nabla \cdot \nabla u \, dA = \iint_D f \, dA
\]

Apply Green’s theorem (essentially the divergence theorem in two dimensions) to the integral on the left side. As a result, the area integral becomes a closed loop integral over the domain’s boundary.

\[
\oint_{\partial D} \nabla u \cdot \hat{n} \, ds = \iint_D f \, dA
\]

The integrand on the left side is also known as the normal derivative, that is, \( \nabla u \cdot \hat{n} = \partial u / \partial n \).

\[
\oint_{\partial D} \frac{\partial u}{\partial n} \, ds = \iint_D f \, dA
\]

The normal derivative is known to be \( g \) on the boundary.

\[
\oint_{\partial D} g \, ds = \iint_D f \, dA
\]

Therefore, for a solution to exist,

\[
\iint_D f \, dx \, dy = \int_{\partial D} g \, ds.
\]

A physical interpretation for this result is that the rate of thermal energy generated in \( D \) must be equal to the rate flowing across the boundary for a steady-state temperature distribution to exist.
In One Dimension

In one dimension the PDE becomes
\[ \frac{d^2 u}{dx^2} = f(x), \quad \text{in } a < x < b, \]
and the boundary conditions associated with it are
\[ \frac{du}{dx} = A \quad \text{at } x = a \]
\[ \frac{du}{dx} = B \quad \text{at } x = b. \]

Integrate both sides of the PDE with respect to \( x \) from \( a \) to \( b \).
\[ \int_{a}^{b} \frac{d^2 u}{dx^2} \, dx = \int_{a}^{b} f(x) \, dx \]
Evaluate the integral on the left side.
\[ \frac{du}{dx}\bigg|_{a}^{b} = \int_{a}^{b} f(x) \, dx \]
Plug in the limits.
\[ \frac{du}{dx}(b) - \frac{du}{dx}(a) = \int_{a}^{b} f(x) \, dx \]
\( du/dx \) is known at the ends.
\[ B - A = \int_{a}^{b} f(x) \, dx \]
Therefore, for a solution to exist,
\[ \int_{a}^{b} f(x) \, dx = B - A. \]
A physical interpretation for this result is that the rate of thermal energy generated in \( D \) must be equal to the rate flowing across the ends for a steady-state temperature distribution to exist.