Exercise 3

Find the solutions that depend only on r of the equation $u_{xx} + u_{yy} = k^2 u$, where k is a positive constant. (*Hint:* Look up Bessel's differential equation in [MF] or in Section 10.5.)

Solution

This PDE is known as the Helmholtz equation.

$$\nabla^2 u = k^2 u$$

Since we're looking for solutions that depend only on r in two dimensions, we choose to write the Laplacian operator in polar coordinates.

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = k^2 u$$

A radially symmetric solution is one that only depends on r, $u = u(r)$. With this assumption the PDE simplifies to an ODE that can be solved relatively easily.

$$\frac{d^2 u}{dr^2} + \frac{1}{r}\frac{du}{dr} = k^2 u$$

Bring $k^2 u$ to the left and multiply both sides by r^2.

$$r^2 \frac{d^2 u}{dr^2} + r\frac{du}{dr} - k^2 r^2 u = 0$$

Write $-k^2$ as $(ik)^2$.

$$r^2 \frac{d^2 u}{dr^2} + r\frac{du}{dr} + (ik)^2 r^2 u = 0$$

This is the parametric form of Bessel's equation of order zero (with parameter ik). The general solution is written in terms of zero-order Bessel functions of the first and second kind, J_0 and Y_0, respectively.

$$u(r) = C_1 J_0(ikr) + C_2 Y_0(ikr)$$

By using different constants and the zero-order modified Bessel functions of the first and second kind, I_0 and K_0, respectively, the arguments can be made real. Therefore,

$$u(r) = AI_0(kr) + BK_0(kr).$$