Exercise 1

Solve $u_{xx} + u_{yy} = 0$ in the rectangle $0 < x < a$, $0 < y < b$ with the following boundary conditions:

$$\begin{array}{llll} u_x = -a & \text{on } x = 0 & u_x = 0 & \text{on } x = a \\ u_y = b & \text{on } y = 0 & u_y = 0 & \text{on } y = b. \end{array}$$

(*Hint:* Note that the necessary condition of Exercise 6.1.11 is satisfied. A shortcut is to guess that the solution might be a quadratic polynomial in x and y.)

Solution

Because the first derivatives of u are constant or zero on the boundary, we seek a solution that is quadratic in x and y.

$$u(x, y) = C_1 + C_2x + C_3x^2 + C_4y + C_5y^2$$

Substitute this form into the PDE and the four boundary conditions to obtain a system of equations involving the constants.

$$\begin{aligned} u_{xx} + u_{yy} = 0 \quad &\rightarrow \quad 2C_3 + 2C_5 = 0 \\ u_x(0, y) = -a \quad &\rightarrow \quad C_2 + 2C_3(0) = -a \\ u_x(a, y) = 0 \quad &\rightarrow \quad C_2 + 2C_3(a) = 0 \\ u_y(x, 0) = b \quad &\rightarrow \quad C_4 + 2C_5(0) = b \\ u_y(x, b) = 0 \quad &\rightarrow \quad C_4 + 2C_5(b) = 0 \end{aligned}$$

Solving the system gives

$$C_2 = -a \quad \text{and} \quad C_3 = \frac{1}{2} \quad \text{and} \quad C_4 = b \quad \text{and} \quad C_5 = -\frac{1}{2}.$$

Therefore,

$$\boxed{u(x, y) = C_1 - ax + \frac{1}{2}x^2 + by - \frac{1}{2}y^2,}$$

where C_1 remains arbitrary.

Note that because the PDE and its boundary conditions are linear, the problem can be split up. Let $u = v + w$, where v and w satisfy the following problems.

$$\begin{array}{ll} \nabla^2 v = 0, \quad 0 < x < a, \ 0 < y < b & \nabla^2 w = 0, \quad 0 < x < a, \ 0 < y < b \\ v_x(0, y) = -a, \quad v_x(a, y) = 0 & w_x(0, y) = 0, \quad w_x(a, y) = 0 \\ v_y(x, 0) = 0, \quad v_y(x, b) = 0 & w_y(x, 0) = b, \quad w_y(x, b) = 0 \end{array}$$

The method of separation of variables can be applied to solve each problem, as all but one boundary condition are homogeneous in each. Doing so yields these series solutions.

$$v(x, y) = A_0 + \sum_{n=1}^{\infty} A_n \cosh\left[\frac{n\pi}{b}(x - a)\right] \cos\frac{n\pi y}{b}$$

$$w(x, y) = B_0 + \sum_{n=1}^{\infty} B_n \cos\frac{m\pi x}{a} \cosh\left[\frac{m\pi}{a}(y - b)\right]$$

The issue with them is that the inhomogeneous boundary conditions cannot be satisfied.