Exercise 2

Prove that the eigenfunctions $\{\sin my \sin nz\}$ are orthogonal on the square $\{0 < y < \pi,\ 0 < z < \pi\}$.

Solution

For the eigenfunctions to be orthogonal on the $\pi \times \pi$ square, they must satisfy

$$\int_0^\pi \int_0^\pi \sin my \sin nz \sin py \sin qz \, dy \, dz = 0,$$

where p and q are integers not equal to m and n, respectively. $(p \neq m,\ q \neq n)$

$$\int_0^\pi \int_0^\pi \sin my \sin nz \sin py \sin qz \, dy \, dz = \left(\int_0^\pi \sin my \sin py \, dy \right) \left(\int_0^\pi \sin nz \sin qz \, dz \right)$$

Apply the product-to-sum formula for sines.

$$\begin{aligned} &= \left\{ \int_0^\pi \frac{1}{2} [\cos(my - py) - \cos(my + py)] \, dy \right\} \\ &\quad \times \left\{ \int_0^\pi \frac{1}{2} [\cos(nz - qz) - \cos(nz + qz)] \, dz \right\} \end{aligned}$$

Split up the integrals.

$$\begin{aligned} &= \frac{1}{2} \left\{ \int_0^\pi \cos[(m-p)y] \, dy - \int_0^\pi \cos[(m+p)y] \, dy \right\} \\ &\quad \times \frac{1}{2} \left\{ \int_0^\pi \cos[(n-q)z] \, dz - \int_0^\pi \cos[(n+q)z] \, dz \right\} \end{aligned}$$

Evaluate the integrals.

$$\begin{aligned} &= \frac{1}{2} \left\{ \frac{1}{m-p} \sin[(m-p)y] \Big|_0^\pi - \frac{1}{m+p} \sin[(m+p)y] \Big|_0^\pi \right\} \\ &\quad \times \frac{1}{2} \left\{ \frac{1}{n-q} \sin[(n-q)z] \Big|_0^\pi - \frac{1}{n+q} \sin[(n+q)z] \Big|_0^\pi \right\} \end{aligned}$$

Substitute the limits. Since $m - p$, $m + p$, $n - q$, and $n + q$ are integers, all the sines are zero.

$$\begin{aligned} &= \frac{1}{2} \left\{ \frac{1}{m-p} \overbrace{\sin[(m-p)\pi]}^{=0} - \frac{1}{m+p} \overbrace{\sin[(m+p)\pi]}^{=0} \right\} \\ &\quad \times \frac{1}{2} \left\{ \frac{1}{n-q} \underbrace{\sin[(n-q)\pi]}_{=0} - \frac{1}{n+q} \underbrace{\sin[(n+q)\pi]}_{=0} \right\} \\ &= 0 \end{aligned}$$

Therefore, the eigenfunctions $\{\sin my \sin nz\}$ are orthogonal on the square $\{0 < y < \pi,\ 0 < z < \pi\}$.