Exercise 3

Find the harmonic function $u(x, y)$ in the square $D = \{0 < x < \pi,\ 0 < y < \pi\}$ with the boundary conditions:

$$u_y = 0 \quad \text{for } y = 0 \text{ and for } y = \pi, \quad u = 0 \quad \text{for } x = 0 \quad \text{and}$$
$$u = \cos^2 y = \tfrac{1}{2}(1 + \cos 2y) \quad \text{for } x = \pi.$$

Solution

A harmonic function $u(x, y)$ is a function that satisfies the Laplace equation, so the boundary value problem we have to solve is the following.

$$\begin{aligned} \nabla^2 u = 0, \quad 0 < x < \pi,\ 0 < y < \pi \\ u_y(x, 0) = 0, \quad u(0, y) = 0 \\ u_y(x, \pi) = 0, \quad u(\pi, y) = \frac{1}{2}(1 + \cos 2y) \end{aligned}$$

As all but one of the boundary conditions are homogeneous, the method of separation of variables can be applied to solve the PDE.

Method 1 - The Hard Way

Assume a product solution of the form $u = X(x)Y(y)$ and plug it into the PDE

$$u_{xx} + u_{yy} = 0 \quad \rightarrow \quad X''Y + XY'' = 0$$

and the homogeneous boundary conditions.

$$\begin{aligned} u_y(x, 0) = 0 \quad &\rightarrow \quad X(x)Y'(0) = 0 \quad \rightarrow \quad Y'(0) = 0 \\ u_y(x, \pi) = 0 \quad &\rightarrow \quad X(x)Y'(\pi) = 0 \quad \rightarrow \quad Y'(\pi) = 0 \\ u(0, y) = 0 \quad &\rightarrow \quad X(0)Y(y) = 0 \quad \rightarrow \quad X(0) = 0 \end{aligned}$$

Now separate variables in the PDE: bring all functions of x to the left side and all functions of y to the right side. Note that the final answer will be the same regardless of which side the minus sign is on.

$$X''Y + XY'' = 0 \quad \rightarrow \quad \frac{X''}{X} = -\frac{Y''}{Y}$$

The only way that a function of x can be equal to a function of y is if both are equal to a constant λ.

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda$$

Values of λ for which the boundary conditions are satisfied are called the eigenvalues, and the nontrivial functions associated with them are called the eigenfunctions.

Determination of Positive Eigenvalues: $\lambda = \mu^2$

If λ is positive, then the ODE for Y becomes

$$-\frac{Y''}{Y} = \mu^2.$$

Multiply both sides by $-Y$.

$$Y'' = -\mu^2 Y$$

The general solution can be written in terms of sine and cosine.

$$Y(y) = C_1 \cos \mu y + C_2 \sin \mu y$$

Take a derivative of it with respect to y.

$$Y'(y) = \mu(-C_1 \sin \mu y + C_2 \cos \mu y)$$

Apply the boundary conditions here to determine C_1 and C_2.

$$\begin{aligned} Y'(0) &= \mu(C_2) = 0 \\ Y'(\pi) &= \mu(-C_1 \sin \mu\pi + C_2 \cos \mu\pi) = 0 \end{aligned}$$

Since $C_2 = 0$, the second equation simplifies to

$$-C_1 \mu \sin \mu\pi = 0.$$

To avoid getting the trivial solution, we insist that $C_1 \neq 0$. Then

$$\sin \mu\pi = 0 \quad \rightarrow \quad \mu\pi = n\pi \quad \rightarrow \quad \mu_n = n, \quad n = 1, 2, \ldots.$$

The eigenfunctions associated with these eigenvalues are

$$Y(y) = C_1 \cos \mu y \quad \rightarrow \quad Y_n(y) = \cos ny, \quad n = 1, 2, \ldots.$$

Now the related ODE for X will be solved.

$$\frac{X''}{X} = \mu^2$$

Multiply both sides by X.

$$X'' = \mu^2 X$$

The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$X(x) = C_3 \cosh \mu x + C_4 \sinh \mu x$$

Apply the boundary condition at $x = 0$ here to determine one of the constants.

$$X(0) = C_3 = 0$$

So then

$$X(x) = C_4 \sinh \mu x \quad \rightarrow \quad X_n(x) = \sinh nx, \quad n = 1, 2, \ldots.$$

Determination of the Zero Eigenvalue: $\lambda = 0$

If λ is zero, then the ODE for Y becomes

$$-\frac{Y''}{Y} = 0.$$

Multiply both sides by $-Y$.

$$Y''(y) = 0$$

Integrate both sides with respect to y.

$$Y'(y) = C_5$$

Apply the boundary conditions here to determine C_5.

$$Y'(0) = C_5 = 0$$
$$Y'(\pi) = C_5 = 0$$

The formula for $Y'(y)$ reduces to

$$Y'(y) = 0$$

Integrate both sides with respect to y once more.

$$Y(y) = C_6$$

Now the related equation for X will be solved.

$$\frac{X''}{X} = 0$$

Multiply both sides by X.

$$X''(x) = 0$$

Integrate both sides with respect to x.

$$X'(x) = C_7$$

Integrate both sides with respect to x once more.

$$X(x) = C_7 x + C_8$$

Apply the boundary condition at $x = 0$ to determine one of the constants.

$$X(0) = C_8 = 0$$

So then

$$X(x) = C_7 x.$$

Determination of Negative Eigenvalues: $\lambda = -\gamma^2$

If λ is negative, then the ODE for Y becomes

$$-\frac{Y''}{Y} = -\gamma^2.$$

Multiply both sides by $-Y$.

$$Y'' = \gamma^2 Y$$

The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$Y(y) = C_9 \cosh \gamma y + C_{10} \sinh \gamma y$$

Take a derivative of it with respect to y.

$$Y'(y) = \gamma(C_9 \sinh \gamma y + C_{10} \cosh \gamma y)$$

Apply the boundary conditions here to determine C_9 and C_{10}.

$$\begin{aligned} Y'(0) &= \gamma(C_{10}) = 0 \\ Y'(\pi) &= \gamma(C_9 \sinh \gamma\pi + C_{10} \cosh \gamma\pi) = 0 \end{aligned}$$

Since $C_{10} = 0$, the second equation reduces to $C_9 \gamma \sinh \gamma\pi = 0$. Hyperbolic sine is not oscillatory, so this equation is only satisfied if $C_9 = 0$. Consequently, the trivial solution, $Y(y) = 0$, is obtained, meaning there are no negative eigenvalues.

According to the principle of superposition, the general solution for u is a linear combination of the eigenfunctions $X(x)Y(y)$ over all the eigenvalues.

$$u(x, y) = A_0 x + \sum_{n=1}^{\infty} A_n \sinh nx \cos ny$$

To determine the coefficients, A_0 and A_n, we use the inhomogeneous boundary condition at $x = \pi$.

$$u(\pi, y) = A_0 \pi + \sum_{n=1}^{\infty} A_n \sinh n\pi \cos ny = \frac{1}{2}(1 + \cos 2y)$$

Because of the form of the right side, A_0 and A_n can be found by matching the coefficients on both sides.

$$A_0 \pi + \sum_{n=1}^{\infty} A_n \sinh n\pi \cos ny = \frac{1}{2} + \frac{1}{2} \cos 2y \quad \rightarrow \quad \begin{cases} A_0 \pi = \dfrac{1}{2} \\ A_2 \sinh 2\pi = \dfrac{1}{2} \\ A_n \sinh n\pi = 0, \quad n \neq 2 \end{cases}$$

Thus, the coefficients are

$$\begin{cases} A_0 = \dfrac{1}{2\pi} \\ A_2 = \dfrac{1}{2 \sinh 2\pi} \\ A_n = 0, \quad n \neq 2 \end{cases}.$$

Therefore,

$$u(x, y) = \frac{1}{2\pi} x + \frac{1}{2 \sinh 2\pi} \sinh 2x \cos 2y.$$

Method 2 - The Easy Way

From the form of the inhomogeneous boundary condition at $x = \pi$,

$$u(\pi, y) = \frac{1}{2} + \frac{1}{2}\cos 2y,$$

we assume the solution has a similar form.

$$u(x, y) = f(x) + g(x)\cos 2y$$

Plug it into the PDE to determine $f(x)$ and $g(x)$.

$$u_{xx} + u_{yy} = f''(x) + g''(x)\cos 2y - 4g(x)\cos 2y = 0$$

Factor $\cos 2y$.

$$f''(x) + [g''(x) - 4g(x)]\cos 2y = 0$$

If we set

$$f''(x) = 0, \tag{1}$$

then the previous equation reduces to

$$[g''(x) - 4g(x)]\cos 2y = 0.$$

Divide both sides by $\cos 2y$ to obtain an ODE for g.

$$g''(x) - 4g(x) = 0 \tag{2}$$

The general solution for f is obtained by integrating twice, and the general solution for g can be written in terms of hyperbolic sine and hyperbolic cosine.

$$\begin{aligned} f(x) &= C_{11}x + C_{12} \\ g(x) &= C_{13}\cosh 2x + C_{14}\sinh 2x \end{aligned}$$

Using the boundary condition at $x = \pi$ gives us two equations.

$$\begin{aligned} f(\pi) &= C_{11}\pi + C_{12} = \frac{1}{2} \\ g(\pi) &= C_{13}\cosh 2\pi + C_{14}\sinh 2\pi = \frac{1}{2} \end{aligned}$$

Using the boundary condition at $x = 0$, $u(0, y) = 0$, gives us two more.

$$\begin{aligned} f(0) &= C_{12} = 0 \\ g(0) &= C_{13} = 0 \end{aligned}$$

With these values for C_{12} and C_{13}, we get

$$C_{11} = \frac{1}{2\pi} \quad \text{and} \quad C_{14} = \frac{1}{2\sinh 2\pi}.$$

So then

$$\begin{aligned} f(x) &= \frac{1}{2\pi}x \\ g(x) &= \frac{1}{2\sinh 2\pi}\sinh 2x. \end{aligned}$$

Therefore,

$$u(x, y) = \frac{1}{2\pi}x + \frac{1}{2\sinh 2\pi}\sinh 2x\cos 2y.$$

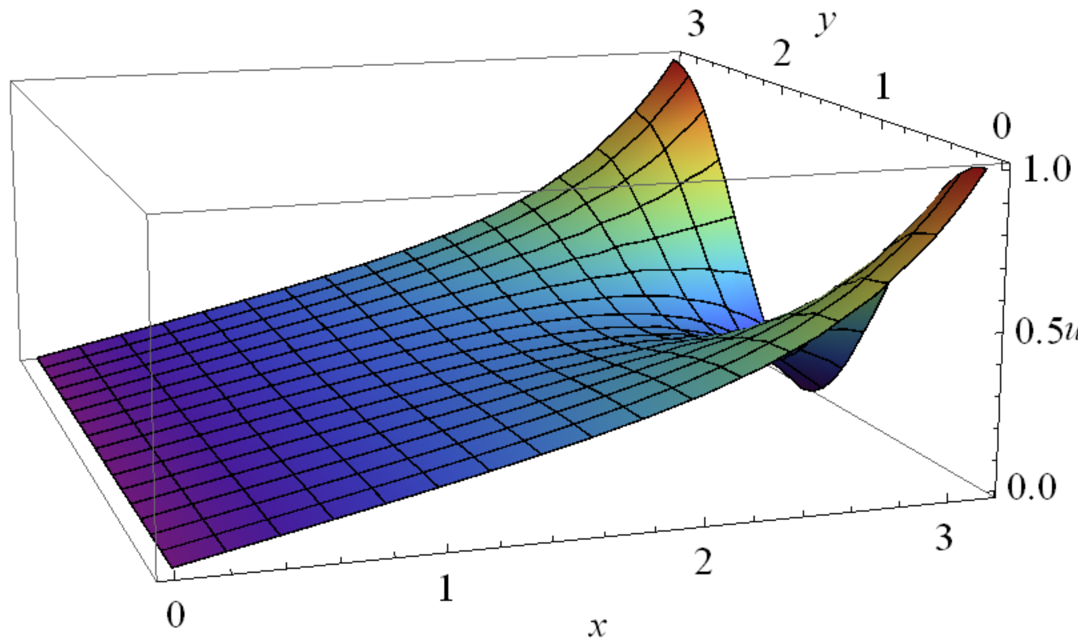

Figure 1: This is a plot of the two-dimensional solution surface $u(x, y)$ in three-dimensional xyu-space. Notice that the maximum and minimum values of u lie on the boundary (maximum principle).