Exercise 4

Find the harmonic function in the square $\{0 < x < 1,\ 0 < y < 1\}$ with the boundary conditions $u(x,0) = x$, $u(x,1) = 0$, $u_x(0,y) = 0$, $u_x(1,y) = y^2$.

Solution

A harmonic function $u(x,y)$ is a function that satisfies the Laplace equation, so the boundary value problem we have to solve is the following.

$$\begin{aligned} \nabla^2 u = 0, &\quad 0 < x < 1,\ 0 < y < 1 \\ u(x,0) = x, &\quad u_x(0,y) = 0 \\ u(x,1) = 0, &\quad u_x(1,y) = y^2 \end{aligned}$$

In order to deal with the two inhomogeneous boundary conditions, we can take advantage of the fact that the PDE is linear. Let $u = v + w$, where v and w satisfy the following problems.

$$\begin{aligned} \nabla^2 v = 0, &\quad 0 < x < 1,\ 0 < y < 1 \\ v(x,0) = x, &\quad v_x(0,y) = 0 \\ v(x,1) = 0, &\quad v_x(1,y) = 0 \end{aligned} \qquad \begin{aligned} \nabla^2 w = 0, &\quad 0 < x < 1,\ 0 < y < 1 \\ w(x,0) = 0, &\quad w_x(0,y) = 0 \\ w(x,1) = 0, &\quad w_x(1,y) = y^2 \end{aligned}$$

The method of separation of variables can be applied to solve each one, as all but one boundary condition are homogeneous in each. The solutions can then be added to obtain u. Assume that v and w have product solutions, $v = X_1(x)Y_1(y)$ and $w = X_2(x)Y_2(y)$, and plug them into their respective PDEs

$$v_{xx} + v_{yy} = 0 \quad \rightarrow \quad X_1''Y_1 + X_1Y_1'' = 0 \tag{1}$$

$$w_{xx} + w_{yy} = 0 \quad \rightarrow \quad X_2''Y_2 + X_2Y_2'' = 0 \tag{2}$$

and homogeneous boundary conditions.

$$\begin{aligned} v(x,1) = 0 &\quad \rightarrow \quad X_1(x)Y_1(1) = 0 \quad \rightarrow \quad Y_1(1) = 0 \\ v_x(0,y) = 0 &\quad \rightarrow \quad X_1'(0)Y_1(y) = 0 \quad \rightarrow \quad X_1'(0) = 0 \\ v_x(1,y) = 0 &\quad \rightarrow \quad X_1'(1)Y_1(y) = 0 \quad \rightarrow \quad X_1'(1) = 0 \\ w(x,0) = 0 &\quad \rightarrow \quad X_2(x)Y_2(0) = 0 \quad \rightarrow \quad Y_2(0) = 0 \\ w(x,1) = 0 &\quad \rightarrow \quad X_2(x)Y_2(1) = 0 \quad \rightarrow \quad Y_2(1) = 0 \\ w_x(0,y) = 0 &\quad \rightarrow \quad X_2'(0)Y_2(y) = 0 \quad \rightarrow \quad X_2'(0) = 0 \end{aligned}$$

Now separate variables in the PDEs: bring all functions of x to the left side and all functions of y to the right side in equations (1) and (2). Note that the final answer for each will be the same regardless of which side the minus sign is on.

$$\begin{aligned} X_1''Y_1 + X_1Y_1'' = 0 &\quad \rightarrow \quad \frac{X_1''}{X_1} = -\frac{Y_1''}{Y_1} \\ X_2''Y_2 + X_2Y_2'' = 0 &\quad \rightarrow \quad \frac{X_2''}{X_2} = -\frac{Y_2''}{Y_2} \end{aligned}$$

The only way that a function of x can be equal to a function of y is if both are equal to a constant. For the two equations we have, these constants are not necessarily the same.

$$\frac{X_1''}{X_1} = -\frac{Y_1''}{Y_1} = \lambda_1$$
$$\frac{X_2''}{X_2} = -\frac{Y_2''}{Y_2} = \lambda_2$$

Values of λ_1 and λ_2 for which the boundary conditions are satisfied are called the eigenvalues, and the nontrivial functions associated with them are called the eigenfunctions.

Determination of Positive Eigenvalues: $\lambda_1 = \mu_1^2$ and $\lambda_2 = \mu_2^2$

If λ_1 and λ_2 are positive, then the ODEs for X_1 and Y_2 become (we solve for X_1 and Y_2 because we have two boundary conditions for each)

$$\frac{X_1''}{X_1} = \mu_1^2 \qquad\qquad -\frac{Y_2''}{Y_2} = \mu_2^2$$
$$X_1'' = \mu_1^2 X_1 \qquad\qquad Y_2'' = -\mu_2^2 Y_2$$
$$X_1(x) = C_1 \cosh\mu_1 x + C_2 \sinh\mu_1 x \qquad\qquad Y_2(y) = C_3 \cos\mu_2 y + C_4 \sin\mu_2 y$$
$$X_1'(x) = \mu_1(C_1 \sinh\mu_1 x + C_2 \cosh\mu_1 x).$$

Apply the boundary conditions, $X_1'(0) = 0$, $X_1'(1) = 0$, $Y_2(0) = 0$, and $Y_2(1) = 0$, here to determine the constants.

$$X_1'(0) = \mu_1(C_2) = 0 \qquad\qquad Y_2(0) = C_3 = 0$$
$$X_1'(1) = \mu_1(C_1 \sinh\mu_1 + C_2 \cosh\mu_1) = 0 \qquad\qquad Y_2(1) = C_3 \cos\mu_2 + C_4 \sin\mu_2 = 0$$

Since $C_2 = 0$ and $C_3 = 0$, the boundary conditions at $x = 1$ and $y = 1$ simplify to

$$X_1'(1) = C_1\mu_1 \sinh\mu_1 = 0 \qquad\qquad Y_2(1) = C_4 \sin\mu_2 = 0.$$

The first equation is only satisfied if $C_1 = 0$ because hyperbolic sine is not oscillatory. To avoid getting the trivial solution, we insist that $C_4 \neq 0$ in the second equation.

$$\sin\mu_2 = 0$$
$$\mu_{2n} = n\pi, \quad n = 1, 2, \ldots$$

The eigenfunctions associated with these eigenvalues for λ_2 are

$$Y_2(y) = C_4 \sin\mu_2 y \quad \rightarrow \quad Y_{2n}(y) = \sin n\pi y, \quad n = 1, 2, \ldots.$$

Since Y_2 is relevant, we now solve the related ODE for X_2.

$$\frac{X_2''}{X_2} = \mu_2^2$$

Multiply both sides by X_2.

$$X_2'' = \mu_2^2 X_2$$

The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$X_2(x) = C_5 \cosh\mu_2 x + C_6 \sinh\mu_2 x$$

Take a derivative of it with respect to x.

$$X_2'(x) = \mu_2(C_5 \sinh \mu_2 x + C_6 \cosh \mu_2 x)$$

Apply the boundary condition at $x = 0$ to determine one of the constants.

$$X_2'(0) = \mu_2(C_6) = 0 \quad \rightarrow \quad C_6 = 0$$

So then

$$X_2(x) = C_5 \cosh \mu_2 x \quad \rightarrow \quad X_{2n}(x) = \cosh n\pi x, \quad n = 1, 2, \ldots.$$

There are no positive eigenvalues for λ_1.

Determination of the Zero Eigenvalue: $\lambda_1 = 0$ and $\lambda_2 = 0$

If λ_1 and λ_2 are zero, then the ODEs for X_1 and Y_2 become (we solve for X_1 and Y_2 because we have two boundary conditions for each)

$$\begin{aligned} \frac{X_1''}{X_1} &= 0 & \qquad -\frac{Y_2''}{Y_2} &= 0 \\ X_1'' &= 0 & Y_2'' &= 0 \\ X_1' &= C_7 & Y_2' &= C_9 \\ X_1(x) &= C_7 x + C_8 & Y_2(y) &= C_9 y + C_{10} \end{aligned}$$

Apply the boundary conditions, $X_1'(0) = 0$, $X_1'(1) = 0$, $Y_2(0) = 0$, and $Y_2(1) = 0$, here to determine the constants.

$$\begin{aligned} X_1'(0) &= C_7 = 0 & \qquad Y_2(0) &= C_{10} = 0 \\ X_1'(1) &= C_7 = 0 & Y_2(1) &= C_9 + C_{10} = 0 \end{aligned}$$

Because $C_{10} = 0$, $C_9 = 0$ as well, and the trivial solution is obtained for Y_2. Zero is only an eigenvalue for λ_1. The eigenfunctions associated with it are constants: $X_1(x) = C_8$. Since X_1 is relevant, we now solve the related ODE for Y_1.

$$-\frac{Y_1''}{Y_1} = 0$$

Multiply both sides by $-Y_1$.

$$Y_1'' = 0$$

The general solution is obtained by integrating both sides with respect to y twice.

$$Y_1(y) = C_{11} y + C_{12}$$

Apply the boundary condition at $y = 1$ to determine one of the constants.

$$Y_1(1) = C_{11} + C_{12} = 0 \quad \rightarrow \quad C_{11} = -C_{12}$$

So then

$$\begin{aligned} Y_1(y) &= -C_{12} y + C_{12} \\ &= C_{12}(1 - y). \end{aligned}$$

Determination of Negative Eigenvalues: $\lambda_1 = -\gamma_1^2$ and $\lambda_2 = -\gamma_2^2$

If λ_1 and λ_2 are negative, then the ODEs for X_1 and Y_2 become (we solve for X_1 and Y_2 because we have two boundary conditions for each)

$$\begin{aligned} \frac{X_1''}{X_1} &= -\gamma_1^2 \\ X_1'' &= -\gamma_1^2 X_1 \\ X_1(x) &= C_{13}\cos\gamma_1 x + C_{14}\sin\gamma_1 x \\ X_1'(x) &= \gamma_1(-C_{13}\sin\gamma_1 x + C_{14}\cos\gamma_1 x). \end{aligned}$$

$$\begin{aligned} -\frac{Y_2''}{Y_2} &= -\gamma_2^2 \\ Y_2'' &= \gamma_2^2 Y_2 \\ Y_2(y) &= C_{15}\cosh\gamma_2 y + C_{16}\sinh\gamma_2 y \end{aligned}$$

Apply the boundary conditions, $X_1'(0) = 0$, $X_1'(1) = 0$, $Y_2(0) = 0$, and $Y_2(1) = 0$, here to determine the constants.

$$\begin{aligned} X_1'(0) &= \gamma_1(C_{14}) = 0 \\ X_1'(1) &= \gamma_1(-C_{13}\sin\gamma_1 + C_{14}\cos\gamma_1) = 0 \end{aligned}$$

$$\begin{aligned} Y_2(0) &= C_{15} = 0 \\ Y_2(1) &= C_{15}\cosh\gamma_2 + C_{16}\sinh\gamma_2 = 0 \end{aligned}$$

Since $C_{14} = 0$ and $C_{15} = 0$, the boundary conditions at $x = 1$ and $y = 1$ simplify to

$$X_1'(1) = -C_{13}\gamma_1\sin\gamma_1 = 0 \qquad Y_2(1) = C_{16}\sinh\gamma_2 = 0.$$

To avoid getting the trivial solution, we insist that $C_{13} \neq 0$ in the first equation. The second equation is only satisfied if $C_{16} = 0$ because hyperbolic sine is not oscillatory.

$$\begin{aligned} \sin\gamma_1 &= 0 \\ \gamma_{1n} &= n\pi, \quad n = 1, 2, \ldots \end{aligned}$$

The eigenfunctions associated with these eigenvalues for λ_1 are

$$X_1(x) = C_{13}\cos\gamma_1 x \quad \rightarrow \quad X_{1n}(x) = \cos n\pi x, \quad n = 1, 2, \ldots.$$

Since X_1 is relevant, we solve the related ODE for Y_1.

$$-\frac{Y_1''}{Y_1} = -\gamma_1^2$$

Multiply both sides by $-Y_1$.

$$Y_1'' = \gamma_1^2 Y_1$$

The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$Y_1(y) = C_{17}\cosh\gamma_1 y + C_{18}\sinh\gamma_1 y$$

Apply the boundary condition at $y = 1$ to determine one of the constants.

$$Y_1(1) = C_{17}\cosh\gamma_1 + C_{18}\sinh\gamma_1 = 0 \quad \rightarrow \quad C_{17} = -C_{18}\frac{\sinh\gamma_1}{\cosh\gamma_1}$$

So then

$$\begin{aligned}
Y_1(y) &= -C_{18}\frac{\sinh\gamma_1}{\cosh\gamma_1}\cosh\gamma_1 y + C_{18}\sinh\gamma_1 y \\
&= -C_{18}\left(\frac{\sinh\gamma_1}{\cosh\gamma_1}\cosh\gamma_1 y - \sinh\gamma_1 y\right) \\
&= -C_{18}\frac{\sinh\gamma_1\cosh\gamma_1 y - \cosh\gamma_1\sinh\gamma_1 y}{\cosh\gamma_1} \\
&= -C_{18}\frac{\sinh(\gamma_1 - \gamma_1 y)}{\cosh\gamma_1} \\
&= C_{19}\sinh[\gamma_1(1-y)] \quad \rightarrow \quad Y_{1n}(y) = \sinh[n\pi(1-y)], \quad n = 1, 2, \ldots .
\end{aligned}$$

There are no negative eigenvalues for λ_2. According to the principle of superposition, the general solutions to the PDEs for v and w are linear combinations of the eigenfunctions, $v = X_1(x)Y_1(y)$ and $w = X_2(x)Y_2(y)$, over all the eigenvalues.

$$\begin{aligned}
v(x,y) &= A_0(1-y) + \sum_{n=1}^{\infty} A_n\cos n\pi x\sinh[n\pi(1-y)] \\
w(x,y) &= \sum_{n=1}^{\infty} B_n\cosh n\pi x\sin n\pi y
\end{aligned}$$

Differentiate the solution for w with respect to x.

$$w_x(x,y) = \sum_{n=1}^{\infty} n\pi B_n\sinh n\pi x\sin n\pi y$$

Set $x = 1$ in the formula for w_x and set $y = 0$ in the formula for v in order to use the inhomogeneous boundary conditions and determine A_0, A_n, and B_n.

$$\begin{aligned}
v(x,0) = A_0 + \sum_{n=1}^{\infty} A_n\cos n\pi x\sinh n\pi = x \\
w_x(1,y) = \sum_{n=1}^{\infty} n\pi B_n\sinh n\pi\sin n\pi y = y^2
\end{aligned}$$

To get A_0, integrate both sides of $v(x,0)$ with respect to x from 0 to 1.

$$\int_0^1\left(A_0 + \sum_{n=1}^{\infty} A_n\cos n\pi x\sinh n\pi\right)dx = \int_0^1 x\,dx$$

Split up the integral on the left into two and bring the constants in front.

$$\int_0^1 A_0\,dx + \sum_{n=1}^{\infty} A_n\sinh n\pi\underbrace{\int_0^1\cos n\pi x\,dx}_{=\,0} = \int_0^1 x\,dx \quad \rightarrow \quad A_0 = \frac{1}{2}$$

To get A_n, multiply both sides of $v(x,0)$ by $\cos m\pi x$, where m is an integer,

$$A_0\cos m\pi x + \sum_{n=1}^{\infty} A_n\sinh n\pi\cos n\pi x\cos m\pi x = x\cos m\pi x$$

and then integrate both sides with respect to x from 0 to 1.

$$\int_0^1 \left(A_0 \cos m\pi x + \sum_{n=1}^{\infty} A_n \sinh n\pi \cos n\pi x \cos m\pi x\right) dx = \int_0^1 x \cos m\pi x \, dx$$

Split up the integral on the left into two and bring the constants in front.

$$A_0 \underbrace{\int_0^1 \cos m\pi x \, dx}_{=0} + \sum_{n=1}^{\infty} A_n \sinh n\pi \int_0^1 \cos n\pi x \cos m\pi x \, dx = \frac{-1 + (-1)^m}{m^2\pi^2}$$

Because the cosine functions are orthogonal, the remaining integral is zero for $n \neq m$. As a result, every term in the infinite series vanishes except for one: $n = m$.

$$A_n \sinh n\pi \int_0^1 \cos^2 n\pi x \, dx = \frac{-1 + (-1)^n}{n^2\pi^2}$$

Evaluate the last integral

$$A_n \sinh n\pi \cdot \frac{1}{2} = \frac{-1 + (-1)^n}{n^2\pi^2}$$

and solve for A_n.

$$A_n = \frac{2}{\sinh n\pi} \frac{-1 + (-1)^n}{n^2\pi^2}$$

To get B_n, multiply both sides of $w_x(1, y)$ by $\sin m\pi y$

$$\sum_{n=1}^{\infty} n\pi B_n \sinh n\pi \sin n\pi y \sin m\pi y = y^2 \sin m\pi y$$

and then integrate both sides with respect to y from 0 to 1.

$$\int_0^1 \sum_{n=1}^{\infty} n\pi B_n \sinh n\pi \sin n\pi y \sin m\pi y \, dy = \int_0^1 y^2 \sin m\pi y \, dy$$

Bring the constants in front of the integral on the left and evaluate the integral on the right.

$$\sum_{n=1}^{\infty} n\pi B_n \sinh n\pi \int_0^1 \sin n\pi y \sin m\pi y \, dy = -\frac{2 + (-1)^m(m^2\pi^2 - 2)}{m^3\pi^3}$$

Because the sine functions are orthogonal, the remaining integral is zero for $n \neq m$. As a result, every term in the infinite series vanishes except for one: $n = m$.

$$n\pi B_n \sinh n\pi \int_0^1 \sin^2 n\pi y \, dy = -\frac{2 + (-1)^n(n^2\pi^2 - 2)}{n^3\pi^3}$$

Evaluate the last integral

$$n\pi B_n \sinh n\pi \cdot \frac{1}{2} = -\frac{2 + (-1)^n(n^2\pi^2 - 2)}{n^3\pi^3}$$

and solve for B_n.

$$B_n = -\frac{2}{n\pi \sinh n\pi} \frac{2 + (-1)^n(n^2\pi^2 - 2)}{n^3\pi^3}$$

Now that the coefficients are known, v and w are known as well.

$$v(x,y) = \frac{1}{2}(1-y) + \sum_{n=1}^{\infty} \frac{2}{\sinh n\pi} \frac{-1+(-1)^n}{n^2\pi^2} \cos n\pi x \sinh[n\pi(1-y)]$$

$$w(x,y) = \sum_{n=1}^{\infty} \frac{-2}{n\pi \sinh n\pi} \frac{2+(-1)^n(n^2\pi^2-2)}{n^3\pi^3} \cosh n\pi x \sin n\pi y$$

Notice that if n is even in the formula for $v(x,y)$, the summand is zero. The result can thus be simplified (that is, made to converge faster) by summing over the odd integers only. Let $n = 2k-1$ in $v(x,y)$ and $n = k$ in $w(x,y)$ to keep the indices of summation the same.

$$v(x,y) = \frac{1}{2}(1-y) + \sum_{2k-1=1}^{\infty} \frac{2}{\sinh[(2k-1)\pi]} \frac{-1+(-1)^{2k-1}}{(2k-1)^2\pi^2} \cos[(2k-1)\pi x] \sinh[(2k-1)\pi(1-y)]$$

$$w(x,y) = \sum_{k=1}^{\infty} \frac{-2}{k\pi \sinh k\pi} \frac{2+(-1)^k(k^2\pi^2-2)}{k^3\pi^3} \cosh k\pi x \sin k\pi y$$

Simplify the formulas.

$$v(x,y) = \frac{1}{2}(1-y) + \sum_{k=1}^{\infty} \frac{-4}{(2k-1)^2\pi^2} \cos[(2k-1)\pi x] \frac{\sinh[(2k-1)\pi(1-y)]}{\sinh[(2k-1)\pi]}$$

$$w(x,y) = -2\sum_{k=1}^{\infty} \frac{2+(-1)^k(k^2\pi^2-2)}{k^4\pi^4} \frac{\cosh k\pi x}{\sinh k\pi} \sin k\pi y$$

Therefore, since $u = v + w$,

$$u(x,y) = \frac{1}{2}(1-y) + \sum_{k=1}^{\infty} \frac{-4}{(2k-1)^2\pi^2} \cos[(2k-1)\pi x] \frac{\sinh[(2k-1)\pi(1-y)]}{\sinh[(2k-1)\pi]} - 2\sum_{k=1}^{\infty} \frac{2+(-1)^k(k^2\pi^2-2)}{k^4\pi^4} \frac{\cosh k\pi x}{\sinh k\pi} \sin k\pi y.$$

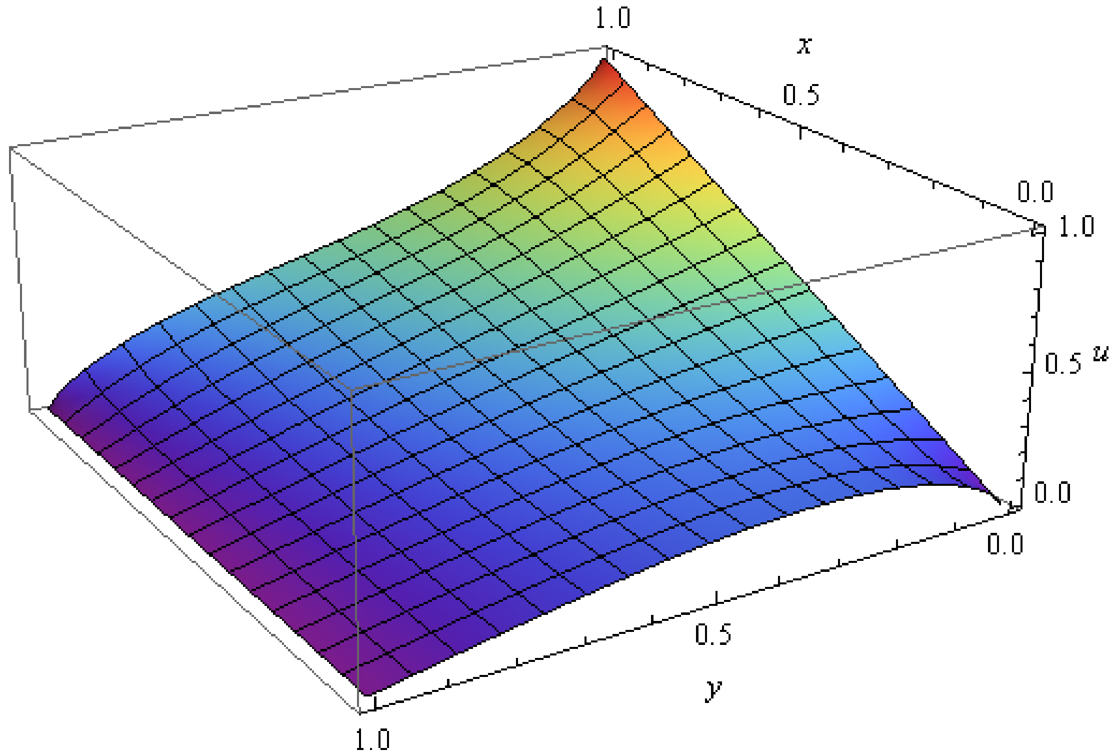

Figure 1: This is a plot of the two-dimensional solution surface $u(x, y)$ in three-dimensional xyu-space. It is approximate, as only the first 100 terms of each infinite series have been used. Notice that the maximum and minimum values of u lie on the boundary (maximum principle).