Exercise 6

Solve the following Neumann problem in the cube $\{0 < x < 1,\ 0 < y < 1,\ 0 < z < 1\}$: $\Delta u = 0$ with $u_z(x, y, 1) = g(x, y)$ and homogeneous Neumann conditions on the other five faces, where $g(x, y)$ is an arbitrary function with zero average.

Solution

The boundary value problem to solve is the following. ($\Delta = \nabla^2$)

$$\begin{gathered}\nabla^2 u = 0, \quad 0 < x < 1,\ 0 < y < 1,\ 0 < z < 1 \\ u_x(0, y, z) = 0, \quad u_y(x, 0, z) = 0, \quad u_z(x, y, 0) = 0 \\ u_x(1, y, z) = 0, \quad u_y(x, 1, z) = 0, \quad u_z(x, y, 1) = g(x, y)\end{gathered}$$

As all but one of the boundary conditions are homogeneous, the method of separation of variables can be applied to solve the PDE. Assume a product solution of the form $u = X(x)Y(y)Z(z)$ and plug it into the PDE

$$u_{xx} + u_{yy} + u_{zz} = 0 \quad \rightarrow \quad X''YZ + XY''Z + XYZ'' = 0$$

and the homogeneous boundary conditions.

$$\begin{aligned}
u_x(0, y, z) = 0 \quad &\rightarrow \quad X'(0)Y(y)Z(z) = 0 \quad &\rightarrow \quad X'(0) = 0 \\
u_x(1, y, z) = 0 \quad &\rightarrow \quad X'(1)Y(y)Z(z) = 0 \quad &\rightarrow \quad X'(1) = 0 \\
u_y(x, 0, z) = 0 \quad &\rightarrow \quad X(x)Y'(0)Z(z) = 0 \quad &\rightarrow \quad Y'(0) = 0 \\
u_y(x, 1, z) = 0 \quad &\rightarrow \quad X(x)Y'(1)Z(z) = 0 \quad &\rightarrow \quad Y'(1) = 0 \\
u_z(x, y, 0) = 0 \quad &\rightarrow \quad X(x)Y(y)Z'(0) = 0 \quad &\rightarrow \quad Z'(0) = 0
\end{aligned}$$

Now separate variables in the PDE: bring all functions of x to the left side and all functions of y and z to the right side. The final answer will be the same regardless of which side the minus sign is on.

$$X''YZ + XY''Z + XYZ'' = 0 \quad \rightarrow \quad \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0 \quad \rightarrow \quad \frac{X''}{X} = -\frac{Y''}{Y} - \frac{Z''}{Z}$$

The only way that a function of x can be equal to a function of y and z is if both are equal to a constant λ.

$$\frac{X''}{X} = -\frac{Y''}{Y} - \frac{Z''}{Z} = \lambda$$

Bring Y''/Y to the right side and λ to the left.

$$-\lambda - \frac{Z''}{Z} = \frac{Y''}{Y}$$

The only way a function of z can be equal to a function of y is if both sides are equal to another constant η.

$$-\lambda - \frac{Z''}{Z} = \frac{Y''}{Y} = \eta$$

To summarize, using the method of separation of variables reduces the Laplace equation to three ODEs, one in each spatial variable.

$$\left.\begin{aligned} \frac{X''}{X} &= \lambda \\ \frac{Y''}{Y} &= \eta \\ -\lambda - \frac{Z''}{Z} &= \eta \end{aligned}\right\}$$

Values of λ and η for which the boundary conditions are satisfied are known as the eigenvalues, and the functions associated with them are known as the eigenfunctions.

Determination of Positive Eigenvalues: $\lambda = \mu^2$ and $\eta = \alpha^2$

If λ and η are positive, then the ODEs in X and Y become

$$\begin{aligned} &\frac{X''}{X} = \mu^2 && \frac{Y''}{Y} = \alpha^2 \\ &X'' = \mu^2 X && Y'' = \alpha^2 Y \\ &X(x) = C_1 \cosh \mu x + C_2 \sinh \mu x && Y(y) = C_3 \cosh \alpha y + C_4 \sinh \alpha y \\ &X'(x) = \mu(C_1 \sinh \mu x + C_2 \cosh \mu x) && Y'(y) = \alpha(C_3 \sinh \alpha y + C_4 \cosh \alpha y). \end{aligned}$$

Apply the boundary conditions here to determine the constants.

$$\begin{aligned} &X'(0) = \mu(C_2) = 0 && Y'(0) = \alpha(C_4) = 0 \\ &X'(1) = \mu(C_1 \sinh \mu + C_2 \cosh \mu) = 0 && Y'(1) = \alpha(C_3 \sinh \alpha + C_4 \cosh \alpha) = 0 \end{aligned}$$

Since $C_2 = 0$ and $C_4 = 0$, the conditions at $x = 1$ and $y = 1$ reduce to

$$C_1 \mu \sinh \mu = 0 \qquad\qquad C_3 \alpha \sinh \alpha = 0.$$

Because hyperbolic sine is not oscillatory, the equations are only satisfied if $C_1 = 0$ and $C_3 = 0$. The trivial solution results for X and Y, so there are no positive eigenvalues for λ and η.

Determination of the Zero Eigenvalue: $\lambda = 0$ and $\eta = 0$

If λ and η are zero, then the ODEs in X and Y become

$$\begin{aligned} &\frac{X''}{X} = 0 && \frac{Y''}{Y} = 0 \\ &X''(x) = 0 && Y''(y) = 0 \\ &X'(x) = C_5 && Y'(y) = C_7 \\ &X(x) = C_5 x + C_6 && Y(y) = C_7 y + C_8 \end{aligned}$$

Apply the boundary conditions here to determine the constants.

$$\begin{aligned} &X'(0) = C_5 = 0 && Y'(0) = C_7 = 0 \\ &X'(1) = C_5 = 0 && Y'(1) = C_7 = 0 \end{aligned}$$

So then

$$X(x) = C_6 \qquad\qquad Y(y) = C_8.$$

If $\lambda = 0$ and $\eta = 0$, then the ODE for Z becomes

$$\frac{Z''}{Z} = 0.$$

Multiply both sides by Z.

$$Z'' = 0$$

Integrate both sides with respect to z.

$$Z'(z) = C_9$$

Apply the boundary condition at $z = 0$ to determine C_9.

$$Z'(0) = C_9 = 0$$

$Z'(z)$ reduces to

$$Z'(z) = 0.$$

Integrate both sides with respect to z once more.

$$Z(z) = C_{10}$$

We find that zero is an eigenvalue for λ and η, and the eigenfunctions associated with them are constants.

Determination of Negative Eigenvalues: $\lambda = -\gamma^2$ and $\eta = -\beta^2$

If λ and η are negative, then the ODEs in X and Y become

$$\begin{aligned} &\frac{X''}{X} = -\gamma^2 && \frac{Y''}{Y} = -\beta^2 \\ &X'' = -\gamma^2 X && Y'' = -\beta^2 Y \\ &X(x) = C_{11}\cos\gamma x + C_{12}\sin\gamma x && Y(y) = C_{13}\cos\beta y + C_{14}\sin\beta y \\ &X'(x) = \gamma(-C_{11}\sin\gamma x + C_{12}\cos\gamma x) && Y'(y) = \beta(-C_{13}\sin\beta y + C_{14}\cos\beta y). \end{aligned}$$

Apply the boundary conditions here to determine the constants.

$$\begin{aligned} &X'(0) = \gamma(C_{12}) = 0 && Y'(0) = \beta(C_{14}) = 0 \\ &X'(1) = \gamma(-C_{11}\sin\gamma + C_{12}\cos\gamma) = 0 && Y'(1) = \beta(-C_{13}\sin\beta + C_{14}\cos\beta) = 0 \end{aligned}$$

Since $C_{12} = 0$ and $C_{14} = 0$, the conditions at $x = 1$ and $y = 1$ reduce to

$$-C_{11}\gamma\sin\gamma = 0 \qquad\qquad -C_{13}\beta\sin\beta = 0.$$

To avoid getting the trivial solution we insist that $C_{11} \neq 0$ and $C_{13} \neq 0$.

$$\begin{aligned} &\sin\gamma = 0 && \sin\beta = 0 \\ &\gamma_m = m\pi, \quad m = 1, 2, \ldots && \beta_n = n\pi, \quad n = 1, 2, \ldots \end{aligned}$$

The eigenfunctions associated with these eigenvalues for λ and η are

$$\begin{aligned} X(x) &= C_{11}\cos\gamma x \quad \rightarrow \quad X_m(x) = \cos m\pi x, \quad m = 1, 2, \ldots \\ Y(y) &= C_{13}\cos\beta y \quad \rightarrow \quad Y_n(y) = \cos n\pi y, \quad n = 1, 2, \ldots \end{aligned}$$

If $\lambda = -m^2\pi^2$ and $\eta = -n^2\pi^2$, then the ODE for Z becomes

$$m^2\pi^2 - \frac{Z''}{Z} = -n^2\pi^2.$$

Solve it for Z''.

$$Z'' = (n^2\pi^2 + m^2\pi^2)Z$$

The solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$Z(z) = C_{15}\cosh(\sqrt{n^2\pi^2 + m^2\pi^2}z) + C_{16}\sinh(\sqrt{n^2\pi^2 + m^2\pi^2}z)$$

Take a derivative of this solution with respect to z.

$$Z'(z) = \sqrt{n^2\pi^2 + m^2\pi^2}\left[C_{15}\sinh(\sqrt{n^2\pi^2 + m^2\pi^2}z) + C_{16}\cosh(\sqrt{n^2\pi^2 + m^2\pi^2}z)\right]$$

Apply the boundary condition at $z = 0$ to determine one of the constants.

$$Z'(0) = \sqrt{n^2\pi^2 + m^2\pi^2}(C_{16}) = 0 \quad \rightarrow \quad C_{16} = 0$$

As a result,

$$Z(z) = C_{15}\cosh(\sqrt{n^2\pi^2 + m^2\pi^2}z) \quad \rightarrow \quad Z_{mn}(z) = \cosh(\pi\sqrt{m^2 + n^2}z), \quad \begin{matrix} m = 1, 2, \ldots \\ n = 1, 2, \ldots \end{matrix}.$$

According to the principle of superposition, the general solution for u is a linear combination of the eigenfunctions $X(x)Y(y)Z(z)$ over all the eigenvalues.

$$\boxed{u(x, y, z) = A_0 + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\cos m\pi x\cos n\pi y\cosh(\pi\sqrt{m^2 + n^2}z)}$$

To determine the coefficients, the boundary condition at $z = 1$ will be applied. Before using it, take a derivative of u with respect to z.

$$u_z(x, y, z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \pi\sqrt{m^2 + n^2}A_{mn}\cos m\pi x\cos n\pi y\sinh(\pi\sqrt{m^2 + n^2}z)$$

Set $z = 1$ in the equation for u_z and use the boundary condition.

$$u_z(x, y, 1) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \pi\sqrt{m^2 + n^2}A_{mn}\cos m\pi x\cos n\pi y\sinh(\pi\sqrt{m^2 + n^2}) = g(x, y)$$

Even though this is technically a double series, it is essentially a Fourier cosine series expansion for $g(x, y)$.

$$\sum_{m=1}^{\infty}\left[\sum_{n=1}^{\infty} \pi\sqrt{m^2 + n^2}A_{mn}\sinh(\pi\sqrt{m^2 + n^2})\cos n\pi y\right]\cos m\pi x = g(x, y)$$

To solve for the term in square brackets, multiply both sides by $\cos p\pi x$, where p is an integer,

$$\sum_{m=1}^{\infty}\left[\sum_{n=1}^{\infty}\pi\sqrt{m^2+n^2}A_{mn}\sinh(\pi\sqrt{m^2+n^2})\cos n\pi y\right]\cos m\pi x\cos p\pi x = g(x,y)\cos p\pi x,$$

and then integrate both sides with respect to x from 0 to 1.

$$\int_0^1\sum_{m=1}^{\infty}\left[\sum_{n=1}^{\infty}\pi\sqrt{m^2+n^2}A_{mn}\sinh(\pi\sqrt{m^2+n^2})\cos n\pi y\right]\cos m\pi x\cos p\pi x\,dx = \int_0^1 g(x,y)\cos p\pi x\,dx$$

Bring the constants in front of the integral on the left.

$$\sum_{m=1}^{\infty}\left[\sum_{n=1}^{\infty}\pi\sqrt{m^2+n^2}A_{mn}\sinh(\pi\sqrt{m^2+n^2})\cos n\pi y\right]\int_0^1\cos m\pi x\cos p\pi x\,dx = \int_0^1 g(x,y)\cos p\pi x\,dx$$

Because the cosine functions are orthogonal, the integral on the left is zero for $m \neq p$. Consequently, every term in the infinite series vanishes except for one: $m = p$.

$$\left[\sum_{n=1}^{\infty}\pi\sqrt{m^2+n^2}A_{mn}\sinh(\pi\sqrt{m^2+n^2})\cos n\pi y\right]\int_0^1\cos^2 m\pi x\,dx = \int_0^1 g(x,y)\cos m\pi x\,dx$$

Evaluate the integral on the left side.

$$\left[\sum_{n=1}^{\infty}\pi\sqrt{m^2+n^2}A_{mn}\sinh(\pi\sqrt{m^2+n^2})\cos n\pi y\right]\cdot\frac{1}{2} = \int_0^1 g(x,y)\cos m\pi x\,dx$$

Multiply both sides by 2.

$$\sum_{n=1}^{\infty}\pi\sqrt{m^2+n^2}A_{mn}\sinh(\pi\sqrt{m^2+n^2})\cos n\pi y = 2\int_0^1 g(x,y)\cos m\pi x\,dx$$

Multiply both sides by $\cos q\pi y$, where q is an integer.

$$\sum_{n=1}^{\infty}\pi\sqrt{m^2+n^2}A_{mn}\sinh(\pi\sqrt{m^2+n^2})\cos n\pi y\cos q\pi y = 2\int_0^1 g(x,y)\cos m\pi x\cos q\pi y\,dx$$

Integrate both sides with respect to y from 0 to 1.

$$\int_0^1\sum_{n=1}^{\infty}\pi\sqrt{m^2+n^2}A_{mn}\sinh(\pi\sqrt{m^2+n^2})\cos n\pi y\cos q\pi y\,dy = \int_0^1 2\int_0^1 g(x,y)\cos m\pi x\cos q\pi y\,dx\,dy$$

Bring the constants in front of the integrals.

$$\sum_{n=1}^{\infty}\pi\sqrt{m^2+n^2}A_{mn}\sinh(\pi\sqrt{m^2+n^2})\int_0^1\cos n\pi y\cos q\pi y\,dy = 2\int_0^1\int_0^1 g(x,y)\cos m\pi x\cos q\pi y\,dx\,dy$$

Because the sine functions are orthogonal, the integral on the left is zero for $n \neq q$. Consequently, every term in the infinite series vanishes except for one: $n = q$.

$$\pi\sqrt{m^2+n^2}A_{mn}\sinh(\pi\sqrt{m^2+n^2})\int_0^1\cos^2 n\pi y\,dy = 2\int_0^1\int_0^1 g(x,y)\cos m\pi x\cos n\pi y\,dx\,dy$$

Evaluate the integral on the left side.

$$\pi\sqrt{m^2+n^2}A_{mn}\sinh(\pi\sqrt{m^2+n^2})\cdot\frac{1}{2} = 2\int_0^1\int_0^1 g(x,y)\cos m\pi x\cos n\pi y\,dx\,dy$$

The equation can now be solved for A_{mn}. Therefore,

$$\boxed{A_{mn} = \frac{4}{\pi\sqrt{m^2+n^2}\sinh(\pi\sqrt{m^2+n^2})}\int_0^1\int_0^1 g(x,y)\cos m\pi x\cos n\pi y\,dx\,dy,}$$

and A_0 remains arbitrary.