Exercise 2

Prove that the eigenfunctions \{\sin(my)\sin(nz)\} are orthogonal on the square \{0 < y < \pi, 0 < z < \pi\}.

Solution

For the eigenfunctions to be orthogonal on the \(\pi \times \pi\) square, they must satisfy

\[
\int_0^\pi \int_0^\pi \sin(my)\sin(nz)\sin(py)\sin(qz)\,dy\,dz = 0,
\]

where \(p\) and \(q\) are integers not equal to \(m\) and \(n\), respectively. \((p \neq m, q \neq n)\)

\[
\int_0^\pi \int_0^\pi \sin(my)\sin(nz)\sin(py)\sin(qz)\,dy\,dz = \left(\int_0^\pi \sin(my)\sin(py)\,dy\right)\left(\int_0^\pi \sin(nz)\sin(qz)\,dz\right)
\]

Apply the product-to-sum formula for sines.

\[
= \left\{\int_0^\pi \frac{1}{2} \left[\cos(my - py) - \cos(my + py)\right]\,dy\right\} \times \left\{\int_0^\pi \frac{1}{2} \left[\cos(nz - qz) - \cos(nz + qz)\right]\,dz\right\}
\]

Split up the integrals.

\[
= \frac{1}{2} \left\{\int_0^\pi \cos((m-p)y)\,dy - \int_0^\pi \cos((m+p)y)\,dy\right\} \times \frac{1}{2} \left\{\int_0^\pi \cos((n-q)z)\,dz - \int_0^\pi \cos((n+q)z)\,dz\right\}
\]

Evaluate the integrals.

\[
= \frac{1}{2} \left\{\frac{1}{m-p} \sin((m-p)y)\right|_0^\pi - \frac{1}{m+p} \sin((m+p)y)\right|_0^\pi\right\} \times \frac{1}{2} \left\{\frac{1}{n-q} \sin((n-q)z)\right|_0^\pi - \frac{1}{n+q} \sin((n+q)z)\right|_0^\pi\right\}
\]

Substitute the limits. Since \(m-p, m+p, n-q,\) and \(n+q\) are integers, all the sines are zero.

\[
= \frac{1}{2} \left\{\frac{1}{m-p} \sin((m-p)\pi) - \frac{1}{m+p} \sin((m+p)\pi)\right\} \times \frac{1}{2} \left\{\frac{1}{n-q} \sin((n-q)\pi) - \frac{1}{n+q} \sin((n+q)\pi)\right\}
\]

\[
= 0
\]

Therefore, the eigenfunctions \{\sin(my)\sin(nz)\} are orthogonal on the square \{0 < y < \pi, 0 < z < \pi\}.