Exercise 3

Find the harmonic function \( u(x, y) \) in the square \( D = \{0 < x < \pi, \ 0 < y < \pi\} \) with the boundary conditions:

\[
\begin{align*}
  u_y &= 0 \quad \text{for } y = 0 \text{ and for } y = \pi, \\
  u &= 0 \quad \text{for } x = 0 \quad \text{and} \\
  u &= \cos^2 y = \frac{1}{2}(1 + \cos 2y) \quad \text{for } x = \pi.
\end{align*}
\]

Solution

A harmonic function \( u(x, y) \) is a function that satisfies the Laplace equation, so the boundary value problem we have to solve is the following.

\[
\nabla^2 u = 0, \quad 0 < x < \pi, \ 0 < y < \pi
\]

\[
 u_y(x, 0) = 0, \quad u(0, y) = 0
\]

\[
 u_y(x, \pi) = 0, \quad u(\pi, y) = \frac{1}{2}(1 + \cos 2y)
\]

As all but one of the boundary conditions are homogeneous, the method of separation of variables can be applied to solve the PDE.

Method 1 - The Hard Way

Assume a product solution of the form \( u = X(x)Y(y) \) and plug it into the PDE

\[
 u_{xx} + u_{yy} = 0 \quad \rightarrow \quad X''Y + XY'' = 0
\]

and the homogeneous boundary conditions.

\[
\begin{align*}
  u_y(x, 0) &= 0 \quad \rightarrow \quad X(x)Y'(0) = 0 \quad \rightarrow \quad Y'(0) = 0 \\
  u_y(x, \pi) &= 0 \quad \rightarrow \quad X(x)Y'(\pi) = 0 \quad \rightarrow \quad Y'(\pi) = 0 \\
  u(0, y) &= 0 \quad \rightarrow \quad X(0)Y(y) = 0 \quad \rightarrow \quad X(0) = 0
\end{align*}
\]

Now separate variables in the PDE: bring all functions of \( x \) to the left side and all functions of \( y \) to the right side. Note that the final answer will be the same regardless of which side the minus sign is on.

\[
X''Y + XY'' = 0 \quad \rightarrow \quad \frac{X''}{X} = -\frac{Y''}{Y}
\]

The only way that a function of \( x \) can be equal to a function of \( y \) is if both are equal to a constant \( \lambda \).

\[
\frac{X''}{X} = -\frac{Y''}{Y} = \lambda
\]

Values of \( \lambda \) for which the boundary conditions are satisfied are called the eigenvalues, and the nontrivial functions associated with them are called the eigenfunctions.

**Determination of Positive Eigenvalues: \( \lambda = \mu^2 \)**

If \( \lambda \) is positive, then the ODE for \( Y \) becomes

\[
-\frac{Y''}{Y} = \mu^2.
\]
Multiply both sides by $-Y$.
\[ Y'' = -\mu^2 Y \]

The general solution can be written in terms of sine and cosine.
\[ Y(y) = C_1 \cos \mu y + C_2 \sin \mu y \]

Take a derivative of it with respect to $y$.
\[ Y'(y) = \mu(-C_1 \sin \mu y + C_2 \cos \mu y) \]

Apply the boundary conditions here to determine $C_1$ and $C_2$.
\[
Y'(0) = \mu(C_2) = 0 \\
Y'(\pi) = \mu(-C_1 \sin \mu \pi + C_2 \cos \mu \pi) = 0
\]

Since $C_2 = 0$, the second equation simplifies to
\[ -C_1 \mu \sin \mu \pi = 0. \]
To avoid getting the trivial solution, we insist that $C_1 \neq 0$. Then
\[ \sin \mu \pi = 0 \rightarrow \mu \pi = n\pi \rightarrow \mu_n = n, \quad n = 1, 2, \ldots. \]

The eigenfunctions associated with these eigenvalues are
\[ Y(y) = C_1 \cos \mu y \rightarrow Y_n(y) = \cos ny, \quad n = 1, 2, \ldots. \]

Now the related ODE for $X$ will be solved.
\[ \frac{X''}{X} = \mu^2 \]

Multiply both sides by $X$.
\[ X'' = \mu^2 X \]

The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.
\[ X(x) = C_3 \cosh \mu x + C_4 \sinh \mu x \]

Apply the boundary condition at $x = 0$ here to determine one of the constants.
\[ X(0) = C_3 = 0 \]

So then
\[ X(x) = C_4 \sinh \mu x \rightarrow X_n(x) = \sinh nx, \quad n = 1, 2, \ldots. \]

**Determination of the Zero Eigenvalue: $\lambda = 0$**

If $\lambda$ is zero, then the ODE for $Y$ becomes
\[ -\frac{Y''}{Y} = 0. \]

Multiply both sides by $-Y$.
\[ Y''(y) = 0 \]

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Integrate both sides with respect to \( y \).
\[
Y'(y) = C_5
\]
Apply the boundary conditions here to determine \( C_5 \).
\[
Y'(0) = C_5 = 0 \\
Y'\left(\pi\right) = C_5 = 0
\]
The formula for \( Y'(y) \) reduces to
\[
Y'(y) = 0
\]
Integrate both sides with respect to \( y \) once more.
\[
Y(y) = C_6
\]
Now the related equation for \( X \) will be solved.
\[
\frac{X''}{X} = 0
\]
Multiply both sides by \( X \).
\[
X''(x) = 0
\]
Integrate both sides with respect to \( x \).
\[
X'(x) = C_7
\]
Integrate both sides with respect to \( x \) once more.
\[
X(x) = C_7x + C_8
\]
Apply the boundary condition at \( x = 0 \) to determine one of the constants.
\[
X(0) = C_8 = 0
\]
So then
\[
X(x) = C_7x.
\]
**Determination of Negative Eigenvalues: \( \lambda = -\gamma^2 \)**

If \( \lambda \) is negative, then the ODE for \( Y \) becomes
\[
-\frac{Y''}{Y} = -\gamma^2.
\]
Multiply both sides by \(-Y\).
\[
Y'' = \gamma^2 Y
\]
The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.
\[
Y(y) = C_9 \cosh \gamma y + C_{10} \sinh \gamma y
\]
Take a derivative of it with respect to \( y \).
\[
Y'(y) = \gamma (C_9 \sinh \gamma y + C_{10} \cosh \gamma y)
\]

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Apply the boundary conditions here to determine $C_9$ and $C_{10}$.

\[
Y'(0) = \gamma(C_{10}) = 0 \\
Y'(\pi) = \gamma(C_9 \sinh \gamma \pi + C_{10} \cosh \gamma \pi) = 0
\]

Since $C_{10} = 0$, the second equation reduces to $C_9 \gamma \sinh \gamma \pi = 0$. Hyperbolic sine is not oscillatory, so this equation is only satisfied if $C_9 = 0$. Consequently, the trivial solution, $Y(y) = 0$, is obtained, meaning there are no negative eigenvalues.

According to the principle of superposition, the general solution for $u$ is a linear combination of the eigenfunctions $X(x)Y(y)$ over all the eigenvalues.

\[
u(x, y) = A_0 x + \sum_{n=1}^{\infty} A_n \sinh nx \cos ny
\]

To determine the coefficients, $A_0$ and $A_n$, we use the inhomogeneous boundary condition at $x = \pi$.

\[
u(\pi, y) = A_0 \pi + \sum_{n=1}^{\infty} A_n \sinh n\pi \cos ny = \frac{1}{2}(1 + \cos 2y)
\]

Because of the form of the right side, $A_0$ and $A_n$ can be found by matching the coefficients on both sides.

\[
A_0 \pi + \sum_{n=1}^{\infty} A_n \sinh n\pi \cos ny = \frac{1}{2} + \frac{1}{2} \cos 2y \quad \Rightarrow \quad \begin{cases} A_0 \pi = \frac{1}{2} \\ A_2 \sinh 2\pi = \frac{1}{2} \\ A_n \sinh n\pi = 0, \quad n \neq 2 \end{cases}
\]

Thus, the coefficients are

\[
\begin{cases} A_0 = \frac{1}{2\pi} \\ A_2 = \frac{1}{2\sinh 2\pi} \\ A_n = 0, \quad n \neq 2 \end{cases}
\]

Therefore,

\[
u(x, y) = \frac{1}{2\pi} x + \frac{1}{2\sinh 2\pi} \sinh 2x \cos 2y.
\]
Method 2 - The Easy Way

From the form of the inhomogeneous boundary condition at $x = \pi$,

$$u(\pi, y) = \frac{1}{2} + \frac{1}{2} \cos 2y,$$

we assume the solution has a similar form.

$$u(x, y) = f(x) + g(x) \cos 2y$$

Plug it into the PDE to determine $f(x)$ and $g(x)$.

$$u_{xx} + u_{yy} = f''(x) + g''(x) \cos 2y - 4g(x) \cos 2y = 0$$

Factor $\cos 2y$.

$$f''(x) + [g''(x) - 4g(x)] \cos 2y = 0$$

If we set

$$f''(x) = 0,$$

then the previous equation reduces to

$$[g''(x) - 4g(x)] \cos 2y = 0.$$  \hspace{1cm} (1)

Divide both sides by $\cos 2y$ to obtain an ODE for $g$.

$$g''(x) - 4g(x) = 0$$  \hspace{1cm} (2)

The general solution for $f$ is obtained by integrating twice, and the general solution for $g$ can be written in terms of hyperbolic sine and hyperbolic cosine.

$$f(x) = C_{11} x + C_{12}$$

$$g(x) = C_{13} \cosh 2x + C_{14} \sinh 2x$$

Using the boundary condition at $x = \pi$ gives us two equations.

$$f(\pi) = C_{11} \pi + C_{12} = \frac{1}{2}$$

$$g(\pi) = C_{13} \cosh 2\pi + C_{14} \sinh 2\pi = \frac{1}{2}$$

Using the boundary condition at $x = 0$, $u(0, y) = 0$, gives us two more.

$$f(0) = C_{12} = 0$$

$$g(0) = C_{13} = 0$$

With these values for $C_{12}$ and $C_{13}$, we get

$C_{11} = \frac{1}{2\pi}$ and $C_{14} = \frac{1}{2 \sinh 2\pi}$.

So then

$$f(x) = \frac{1}{2\pi} x$$

$$g(x) = \frac{1}{2 \sinh 2\pi} \sinh 2x.$$  \hspace{1cm} (3)

Therefore,

$$u(x, y) = \frac{1}{2\pi} x + \frac{1}{2 \sinh 2\pi} \sinh 2x \cos 2y.$$
Figure 1: This is a plot of the two-dimensional solution surface $u(x, y)$ in three-dimensional $xyu$-space. Notice that the maximum and minimum values of $u$ lie on the boundary (maximum principle).