Exercise 2

Solve $u_{xx} + u_{yy} = 0$ in the disk $\{r < a\}$ with the boundary condition

$$u = 1 + 3\sin\theta \quad \text{on } r = a.$$

Solution

The PDE to solve is Laplace's equation.

$$\nabla^2 u = 0$$

Since the boundary condition is given on a circle of radius a, we opt to write the Laplacian operator ∇^2 in polar coordinates and solve for u as a function of r and θ.

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \tag{1}$$

From the form of the boundary condition we hypothesize that the solution has the form

$$u(r,\theta) = 1 + f(r)\sin\theta.$$

In order to determine $f(r)$, substitute this expression for $u(r,\theta)$ into equation (1).

$$\frac{\partial^2}{\partial r^2}[1 + f(r)\sin\theta] + \frac{1}{r}\frac{\partial}{\partial r}[1 + f(r)\sin\theta] + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}[1 + f(r)\sin\theta] = 0$$

Evaluate the derivatives.

$$f''(r)\sin\theta + \frac{1}{r}f'(r)\sin\theta + \frac{1}{r^2}f(r)(-\sin\theta) = 0$$

Multiply both sides by $r^2/\sin\theta$.

$$r^2 f'' + rf' - f = 0$$

Since θ is not present in this equation, the hypothesis for $u(r,\theta)$ is legitimate. This is an equidimensional ODE for f, so it has solutions of the form

$$f(r) = r^m \quad \rightarrow \quad f'(r) = mr^{m-1} \quad \rightarrow \quad f''(r) = m(m-1)r^{m-2}.$$

Substitute these expressions into the ODE to determine the constants m.

$$m(m-1)r^m + mr^m - r^m = 0$$

Divide both sides by r^m.

$$m(m-1) + m - 1 = 0$$

Solve for m.

$$m^2 - 1 = 0 \quad \rightarrow \quad m = \{\pm 1\}$$

Consequently,

$$f(r) = C_1 r + C_2 r^{-1}.$$

C_1 and C_2 are determined by using the boundary conditions, $f(a) = 3$ and $f(0) = \text{finite}$.

$$f(a) = C_1 a + \frac{C_2}{a} = 3$$

$$f(0) = \text{finite} \quad \rightarrow \quad C_2 = 0$$

Because $C_2 = 0$, the first equation reduces to $C_1 a = 3$, so $C_1 = 3/a$.

$$f(r) = \frac{3r}{a}$$

The solution for u in polar coordinates is then

$$u(r,\theta) = 1 + \frac{3r}{a}\sin\theta.$$

Since $y = r\sin\theta$, the solution in Cartesian coordinates is therefore

$$u(x,y) = 1 + \frac{3y}{a}.$$

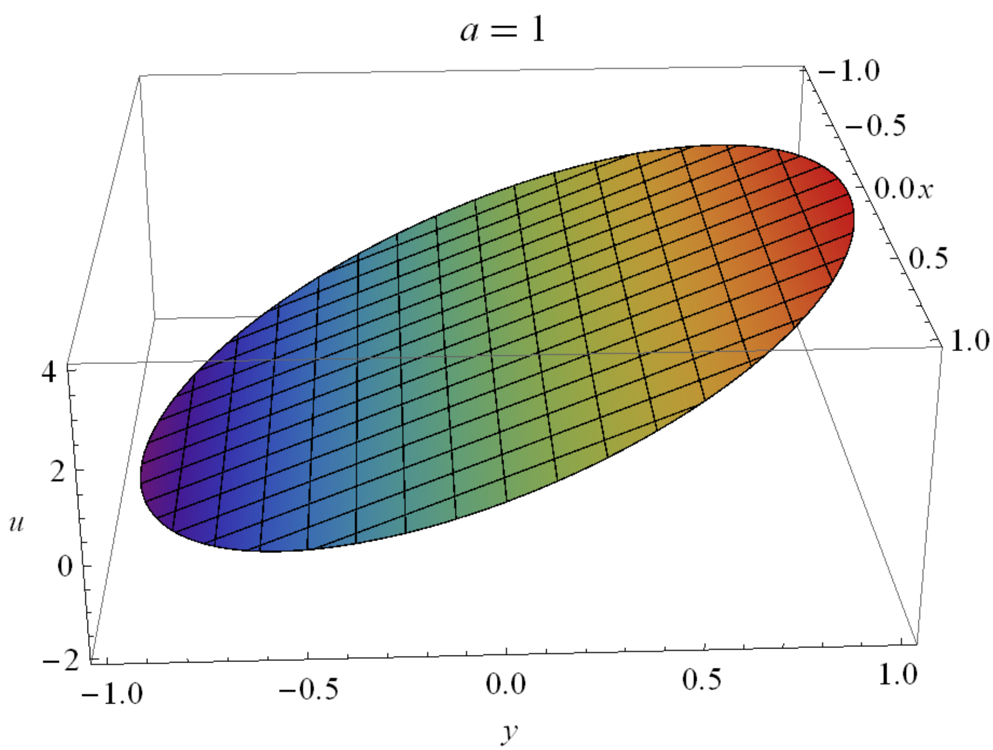

Figure 1: This is a plot of the two-dimensional solution surface $u(x,y)$ in three-dimensional xyu-space for $a = 1$. Notice that the maximum and minimum values of u lie on the boundary (maximum principle) and that the value of u at the origin is the average of values along the boundary (mean value property).