Exercise 1

Suppose that $u$ is a harmonic function in the disk $D = \{r < 2\}$ and that $u = 3\sin 2\theta + 1$ for $r = 2$. Without finding the solution, answer the following questions.

(a) Find the maximum value of $u$ in $D$.

(b) Calculate the value of $u$ at the origin.

Solution

The fact that $u$ is a harmonic function means that it satisfies Laplace's equation.

$$\nabla^2 u = 0$$

Part (a)

$D$ is the union of $D$ and its boundary: $\overline{D} = D \cup \text{bdy } D = \{r \leq 2\}$. According to the maximum principle, the maximum value of $u$ in $\overline{D}$ occurs on the boundary $r = 2$. Our task then is to find the maximum of the prescribed function $u(2, \theta) = 3\sin 2\theta + 1$.

$$\frac{d}{d\theta} (3\sin 2\theta + 1) = 6\cos 2\theta = 0 \quad \rightarrow \quad 2\theta = \left\{ \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2} \right\}$$

The critical values of $\theta$ are $\pi/4$, $3\pi/4$, $5\pi/4$, and $7\pi/4$.

$$u\left(2, \frac{\pi}{4}\right) = 4$$

$$u\left(2, \frac{3\pi}{4}\right) = -2$$

$$u\left(2, \frac{5\pi}{4}\right) = 4$$

$$u\left(2, \frac{7\pi}{4}\right) = -2$$

Therefore, the maximum value of $u$ in $\overline{D}$ is 4.

Part (b)

According to the mean value property, the value of $u$ at the origin is the average of $u$ on the circumference. What we have to do then is integrate the prescribed function $u(2, \theta)$ over the circumference and then divide the result by the circumference.

$$u(0) = \frac{\int_0^{2\pi} u(2, \theta) \, d\theta}{\int_0^{2\pi} ds} = \frac{\int_0^{2\pi} u(2, \theta)(2 \, d\theta)}{\int_0^{2\pi} 2 \, d\theta} = \frac{1}{2\pi} \int_0^{2\pi} u(2, \theta) \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} (3\sin 2\theta + 1) \, d\theta$$

$$= \frac{1}{2\pi} \left( -\frac{3}{2} \cos 2\theta + \theta \right) \bigg|_0^{2\pi} = 1$$

Therefore, the value of $u$ at the origin is 1.