Exercise 10

Solve $u_{xx} + u_{yy} = 0$ in the quarter-disk $\{x^2 + y^2 < a^2,\ x > 0,\ y > 0\}$ with the following BCs:

$$u = 0 \quad \text{on } x = 0 \text{ and on } y = 0 \quad \text{and} \quad \frac{\partial u}{\partial r} = 1 \quad \text{on } r = a.$$

Write the answer as an infinite series and write the first two nonzero terms explicitly.

Solution

Since the domain is a quarter-disk, we choose to expand the Laplacian operator in polar coordinates. The boundary value problem to solve then is the following.

$$\begin{gathered} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad r < a, 0 < \theta < \frac{\pi}{2} \\ u(0,\theta) = \text{bounded}, \quad u(r,0) = 0 \\ u_r(a,\theta) = 1, \quad u\left(r, \frac{\pi}{2}\right) = 0 \end{gathered}$$

The Laplace equation and all but one of its associated boundary conditions are linear and homogeneous, so the method of separation of variables can be applied to solve the problem. Assume a product solution of the form $u = R(r)\Theta(\theta)$ and plug it into the PDE

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \quad \rightarrow \quad R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0$$

and the homogeneous boundary conditions.

$$\begin{aligned} u(r,0) = 0 \quad &\rightarrow \quad R(r)\Theta(0) = 0 \quad &\rightarrow \quad \Theta(0) = 0 \\ u\left(r,\frac{\pi}{2}\right) = 0 \quad &\rightarrow \quad R(r)\Theta\left(\frac{\pi}{2}\right) = 0 \quad &\rightarrow \quad \Theta\left(\frac{\pi}{2}\right) = 0 \\ u(0,\theta) = \text{bounded} \quad &\rightarrow \quad R(0)\Theta(\theta) = \text{bounded} \quad &\rightarrow \quad R(0) = \text{bounded} \end{aligned}$$

Now separate variables in the PDE: bring all functions of r to the left side and all functions of θ to the right side. Note that the final answer will be the same regardless of which side the minus sign is on.

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0 \quad \rightarrow \quad r^2\frac{R''}{R} + r\frac{R'}{R} + \frac{\Theta''}{\Theta} = 0 \quad \rightarrow \quad r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta}$$

The only way that a function of r can be equal to a function of θ is if both are equal to a constant λ.

$$r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = \lambda$$

Values of λ for which the boundary conditions are satisfied are called the eigenvalues, and the nontrivial functions associated with them are called the eigenfunctions.

Determination of Positive Eigenvalues: $\lambda = \mu^2$

If λ is positive, then the ODE for Θ becomes

$$-\frac{\Theta''}{\Theta} = \mu^2.$$

Multiply both sides by $-\Theta$.

$$\Theta'' = -\mu^2\Theta$$

The general solution can be written in terms of sine and cosine.

$$\Theta(\theta) = C_1 \cos\mu\theta + C_2 \sin\mu\theta$$

Apply the boundary conditions here to determine C_1 and C_2.

$$\Theta(0) = C_1 = 0$$
$$\Theta\left(\frac{\pi}{2}\right) = C_1 \cos\left(\mu\frac{\pi}{2}\right) + C_2 \sin\left(\mu\frac{\pi}{2}\right) = 0$$

Since $C_1 = 0$, the second equation reduces to

$$C_2 \sin\left(\mu\frac{\pi}{2}\right) = 0.$$

To avoid getting the trivial solution, we insist that $C_2 \neq 0$. Then

$$\sin\left(\mu\frac{\pi}{2}\right) = 0,$$

and $\mu\pi/2$ must be an integer multiple of π.

$$\mu\frac{\pi}{2} = n\pi$$
$$\mu_n = 2n, \quad n = 1, 2, \ldots$$

The eigenfunctions corresponding with these eigenvalues for λ are

$$\Theta(\theta) = C_2 \sin\mu\theta \quad \rightarrow \quad \Theta_n(\theta) = \sin 2n\theta, \quad n = 1, 2, \ldots.$$

The ODE for $R(r)$ will now be solved.

$$r^2\frac{R''}{R} + r\frac{R'}{R} = \mu^2$$

Bring μ^2 to the left side and multiply both sides by R.

$$r^2R'' + rR' - \mu^2R = 0$$

This is an equidimensional ODE, so the solutions are of the form, $R(r) = r^m$. Find the first and second derivatives of it

$$R = r^m \quad \rightarrow \quad R' = mr^{m-1} \quad \rightarrow \quad R'' = m(m-1)r^{m-2}$$

and then substitute these formulas into the ODE to determine the values of m.

$$m(m-1)r^m + mr^m - \mu^2r^m = 0$$

$$m(m-1) + m - \mu^2 = 0$$

$$m^2 - \mu^2 = 0$$

$$m = \{\pm\mu\}$$

So then

$$R(r) = C_3 r^{\mu} + C_4 r^{-\mu}.$$

For the boundary condition, $R(0) =$ bounded, to be satisfied, we require $C_4 = 0$.

$$R(r) = C_3 r^{\mu} \quad \rightarrow \quad R_n(r) = r^{2n}, \quad n = 1, 2, \ldots$$

Determination of the Zero Eigenvalue: $\lambda = 0$

If λ is zero, then the ODE for Θ becomes

$$-\frac{\Theta''}{\Theta} = 0.$$

Multiply both sides by $-\Theta$.

$$\Theta''(\theta) = 0$$

Integrate both sides with respect to θ twice to solve for Θ.

$$\Theta(\theta) = C_5\theta + C_6$$

Apply the boundary conditions here to determine C_5 and C_6.

$$\Theta(0) = C_6 = 0$$
$$\Theta\left(\frac{\pi}{2}\right) = C_5\frac{\pi}{2} + C_6 = 0$$

Since $C_6 = 0$, the second equation yields $C_5 = 0$, resulting in the trivial solution. Consequently, zero is not an eigenvalue.

Determination of Negative Eigenvalues: $\lambda = -\gamma^2$

If λ is negative, then the ODE for Θ becomes

$$-\frac{\Theta''}{\Theta} = -\gamma^2.$$

Multiply both sides by $-\Theta$.

$$\Theta'' = \gamma^2\Theta$$

The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$\Theta(\theta) = C_7 \cosh\gamma\theta + C_8 \sinh\gamma\theta$$

Apply the boundary conditions here to find C_7 and C_8.

$$\Theta(0) = C_7 = 0$$
$$\Theta\left(\frac{\pi}{2}\right) = C_7 \cosh\left(\gamma\frac{\pi}{2}\right) + C_8 \sinh\left(\gamma\frac{\pi}{2}\right) = 0$$

Since $C_7 = 0$, the second equation reduces to $C_8 \sinh\gamma\pi/2 = 0$. Hyperbolic sine is not oscillatory, so the only way the second equation is satisfied is if $C_8 = 0$. The trivial solution is obtained, so there are no negative eigenvalues.

According to the principle of superposition, the general solution for u is a linear combination of the eigenfunctions $R(r)\Theta(\theta)$ over all the eigenvalues.

$$u(r,\theta) = \sum_{n=1}^{\infty} B_n r^{2n} \sin 2n\theta$$

The coefficients B_n are determined by applying the inhomogeneous boundary condition at $r = a$. Take a derivative of the solution with respect to r

$$u_r(r,\theta) = \sum_{n=1}^{\infty} 2nB_n r^{2n-1} \sin 2n\theta$$

and set $r = a$ in order to use it.

$$u_r(a,\theta) = \sum_{n=1}^{\infty} 2nB_n a^{2n-1} \sin 2n\theta = 1$$

Multiply both sides by $\sin 2p\theta$, where p is an integer.

$$\sum_{n=1}^{\infty} 2nB_n a^{2n-1} \sin 2n\theta \sin 2p\theta = \sin 2p\theta$$

Integrate both sides with respect to θ from 0 to $\pi/2$.

$$\int_0^{\pi/2} \sum_{n=1}^{\infty} 2nB_n a^{2n-1} \sin 2n\theta \sin 2p\theta \, d\theta = \int_0^{\pi/2} \sin 2p\theta \, d\theta$$

Bring the constants in front of the integral on the left side.

$$\sum_{n=1}^{\infty} 2nB_n a^{2n-1} \int_0^{\pi/2} \sin 2n\theta \sin 2p\theta \, d\theta = \int_0^{\pi/2} \sin 2p\theta \, d\theta$$

Because the sine functions are orthogonal, the integral on the left is zero for $n \neq p$. As a result, every term in the infinite series vanishes except for one: $n = p$.

$$2nB_n a^{2n-1} \int_0^{\pi/2} \sin^2 2n\theta \, d\theta = \int_0^{\pi/2} \sin 2n\theta \, d\theta$$

Evaluate the integrals.

$$\begin{aligned} 2nB_n a^{2n-1} \cdot \frac{\pi}{4} &= \frac{1}{2n}(-\cos 2n\theta)\bigg|_0^{\pi/2} \\ &= \frac{1-\cos n\pi}{2n} \\ &= \frac{1-(-1)^n}{2n} \end{aligned}$$

Solve this equation for B_n.

$$B_n = \frac{1}{n\pi a^{2n-1}} \frac{1-(-1)^n}{n}$$

So then

$$u(r,\theta) = \sum_{n=1}^{\infty} \frac{1}{n\pi a^{2n-1}} \frac{1-(-1)^n}{n} r^{2n} \sin 2n\theta.$$

Notice that the summand is zero if n is even. The solution can thus be simplified (that is, made to converge faster) by summing over the odd integers only. Let $n = 2k-1$.

$$u(r,\theta) = \sum_{2k-1=1}^{\infty} \frac{1}{(2k-1)\pi a^{2(2k-1)-1}} \frac{1-(-1)^{2k-1}}{2k-1} r^{2(2k-1)} \sin[2(2k-1)\theta]$$

Therefore,

$$\begin{aligned} u(r,\theta) &= \sum_{k=1}^{\infty} \frac{1}{(2k-1)\pi a^{4k-3}} \frac{2}{2k-1} r^{2(2k-1)} \sin[2(2k-1)\theta] \\ &= \frac{2}{\pi a} r^2 \sin 2\theta + \frac{2}{9\pi a^5} r^6 \sin 6\theta + \cdots . \end{aligned}$$

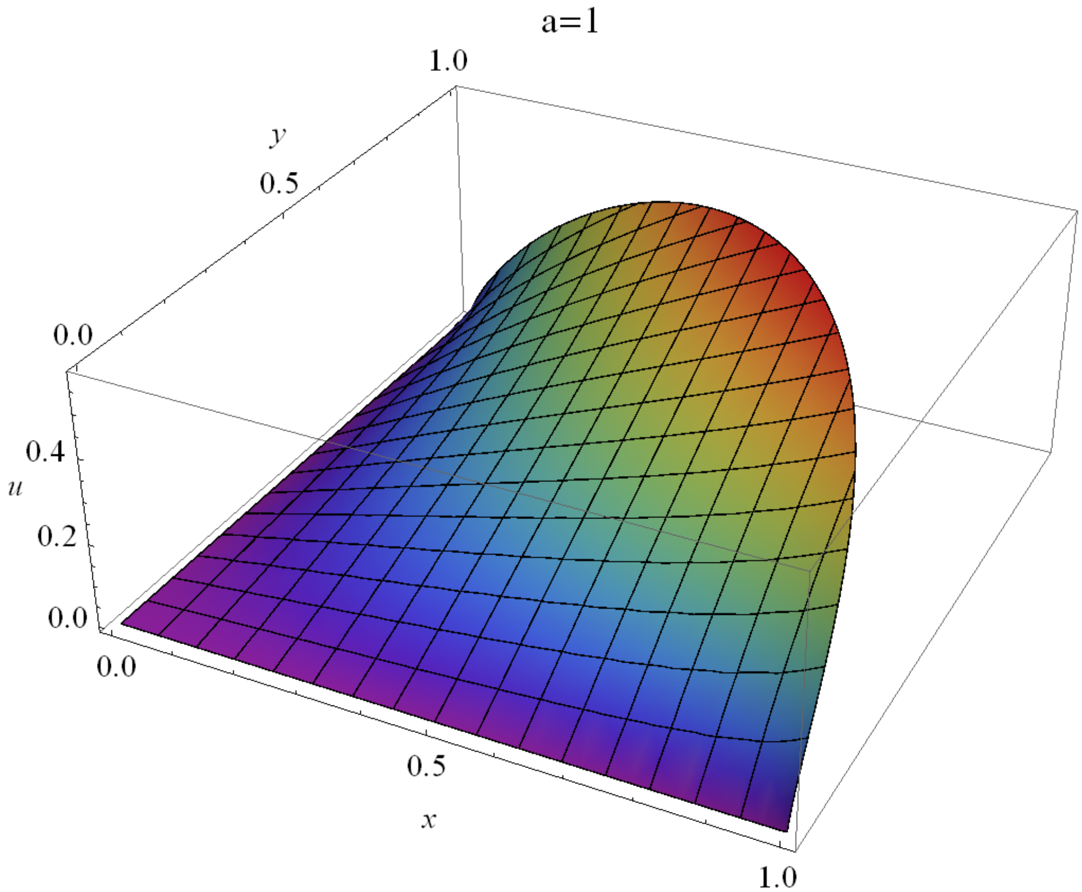

Figure 1: This is a plot of the two-dimensional solution surface $u(x,y)$ in three-dimensional xyu-space for $a=1$. Notice that the maximum and minimum values of u lie on the boundary (maximum principle).