Exercise 11

Prove the uniqueness of the Robin problem

$$\Delta u = f \quad \text{in } D, \quad \frac{\partial u}{\partial n} + au = h \quad \text{on bdy } D,$$

where D is any domain in three dimensions and where a is a positive constant.

Solution

Suppose that u and v are two solutions of the Robin problem. Then

$$\begin{aligned} \Delta u = f \quad \text{in } D, \quad \frac{\partial u}{\partial n} + au = h \quad \text{on bdy } D \\ \Delta v = f \quad \text{in } D, \quad \frac{\partial v}{\partial n} + av = h \quad \text{on bdy } D. \end{aligned}$$

Subtract both sides of each equation on the bottom from those of each equation on top.

$$\begin{aligned} \Delta u - \Delta v = f - f \quad \text{in } D, \quad \frac{\partial u}{\partial n} + au - \left(\frac{\partial v}{\partial n} + av\right) = h - h \quad \text{on bdy } D \\ \Delta(u - v) = 0 \quad \text{in } D, \quad \frac{\partial}{\partial n}(u - v) + a(u - v) = 0 \quad \text{on bdy } D \end{aligned}$$

Let $w = u - v$.

$$\Delta w = 0 \quad \text{in } D, \quad \frac{\partial w}{\partial n} + aw = 0 \quad \text{on bdy } D$$

Multiply both sides of the equation on the left by w and bring aw to the right side of the equation on the right.

$$w\Delta w = 0 \quad \text{in } D, \quad \frac{\partial w}{\partial n} = -aw \quad \text{on bdy } D$$

Integrate both sides of the equation on the left over the volume of the domain.

$$\iiint_D w\Delta w \, dV = 0 \quad \text{in } D$$

Apply Green's first identity (page 180) to the volume integral.

$$\iint_{\text{bdy } D} w\frac{\partial w}{\partial n} \, dS - \iiint_D |\nabla w|^2 \, dV = 0 \quad \text{in } D$$

Apply the boundary condition to the surface integral in the equation on the left.

$$\iint_{\text{bdy } D} w(-aw) \, dS - \iiint_D |\nabla w|^2 \, dV = 0$$

Bring $-a$ in front of the surface integral and bring the volume integral to the right side.

$$-a \iint_{\text{bdy } D} w^2 \, dS = \iiint_D |\nabla w|^2 \, dV$$

Because both of the integrands are positive, both of the integrals must be positive as well. a is assumed to be positive, so a negative number is equal to a positive number. Both sides must be equal to zero.

$$-a \iint\limits_{\text{bdy } D} w^2 \, dS = \iiint_D |\nabla w|^2 \, dV = 0$$

Consequently,

$$\begin{aligned} |\nabla w|^2 = 0 \quad \text{in } D, \qquad w^2 = 0 \quad \text{on bdy } D \\ |\nabla w| = 0 \quad \text{in } D, \qquad w = 0 \quad \text{on bdy } D \\ \sqrt{\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2} = 0 \quad \text{in } D \\ w = C_1 \quad \text{in } D. \end{aligned}$$

Since $w = 0$ on bdy D, we conclude that $w = 0$ in D as well.

$$\begin{aligned} u - v = 0 \\ u = v \end{aligned}$$

We find that the two solutions to the Robin problem are one and the same. Therefore, the solution to the Robin problem is unique.