Exercise 6

Find the harmonic function u in the semidisk $\{r < 1,\ 0 < \theta < \pi\}$ with u vanishing on the diameter $(\theta = 0, \pi)$ and

$$u = \pi \sin\theta - \sin 2\theta \quad \text{on } r = 1.$$

Solution

A harmonic function u is a function that satisfies the Laplace equation.

$$\nabla^2 u = 0$$

Since the domain is a semidisk, the Laplacian operator will be expanded in polar coordinates. Together with the boundary conditions, the problem we have to solve is the following.

$$\begin{gathered} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad r < 1,\ 0 < \theta < \pi \\ u(0,\theta) = \text{bounded}, \quad u(r,0) = 0 \\ u(1,\theta) = \pi\sin\theta - \sin 2\theta, \quad u(r,\pi) = 0 \end{gathered}$$

We hypothesize from the form of the inhomogeneous boundary condition that the solution is

$$u(r,\theta) = f(r)\sin\theta + g(r)\sin 2\theta.$$

Apply the boundary conditions for u to determine the boundary conditions for f and g.

$$\begin{aligned} u(0,\theta) &= f(0)\sin\theta + g(0)\sin 2\theta = \text{bounded} && \to \quad f(0) = \text{bounded} \quad \text{and} \quad g(0) = \text{bounded} \\ u(1,\theta) &= f(1)\sin\theta + g(1)\sin 2\theta = \pi\sin\theta - \sin 2\theta && \to \quad f(1) = \pi \quad \text{and} \quad g(1) = -1 \end{aligned}$$

In order to determine $f(r)$ and $g(r)$, substitute the expression for $u(r,\theta)$ into the PDE.

$$\frac{\partial^2}{\partial r^2}[f(r)\sin\theta + g(r)\sin 2\theta] + \frac{1}{r}\frac{\partial}{\partial r}[f(r)\sin\theta + g(r)\sin 2\theta] + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}[f(r)\sin\theta + g(r)\sin 2\theta] = 0$$

Evaluate the derivatives.

$$f''(r)\sin\theta + g''(r)\sin 2\theta + \frac{1}{r}[f'(r)\sin\theta + g'(r)\sin 2\theta] + \frac{1}{r^2}[-f(r)\sin\theta - 4g(r)\sin 2\theta] = 0$$

Expand the left side.

$$f''(r)\sin\theta + \frac{1}{r}f'(r)\sin\theta - \frac{1}{r^2}f(r)\sin\theta + g''(r)\sin 2\theta + \frac{1}{r}g'(r)\sin 2\theta - \frac{4}{r^2}g(r)\sin 2\theta = 0$$

If we set

$$f''(r)\sin\theta + \frac{1}{r}f'(r)\sin\theta - \frac{1}{r^2}f(r)\sin\theta = 0, \tag{1}$$

then the previous equation reduces to

$$g''(r)\sin 2\theta + \frac{1}{r}g'(r)\sin 2\theta - \frac{4}{r^2}g(r)\sin 2\theta = 0. \tag{2}$$

Multiply both sides of equation (1) by $r^2/\sin\theta$ and both sides of equation (2) by $r^2/\sin 2\theta$.

$$r^2 f'' + r f' - f = 0 \tag{3}$$

$$r^2 g'' + r g' - 4g = 0 \tag{4}$$

These equations for f and g are independent of θ, so the hypothesis for $u(r,\theta)$ is legitimate. Both are equidimensional ODEs and have solutions of the form, $f(r) = r^m$ and $g(r) = r^n$. Plug these forms into equations (3) and (4) to find the appropriate values of m and n.

$$f(r) = r^m \quad \to \quad f'(r) = mr^{m-1} \quad \to \quad f''(r) = m(m-1)r^{m-2}$$

$$m(m-1)r^m + mr^m - r^m = 0$$

$$m(m-1) + m - 1 = 0$$

$$m^2 - 1 = 0$$

$$m = \{\pm 1\}$$

$$g(r) = r^n \quad \to \quad g'(r) = nr^{n-1} \quad \to \quad g''(r) = n(n-1)r^{n-2}$$

$$n(n-1)r^n + nr^n - 4r^n = 0$$

$$n(n-1) + n - 4 = 0$$

$$n^2 - 4 = 0$$

$$n = \{\pm 2\}$$

Consequently,

$$\begin{aligned} f(r) &= C_1 r + C_2 r^{-1} \\ g(r) &= C_3 r^2 + C_4 r^{-2}. \end{aligned}$$

To satisfy $f(0) =$ bounded and $g(0) =$ bounded, we require $C_2 = 0$ and $C_4 = 0$. Apply the boundary conditions at $r = 1$ to determine C_1 and C_3.

$$\begin{aligned} f(1) &= C_1 = \pi \\ g(1) &= C_3 = -1 \end{aligned}$$

So then

$$\begin{aligned} f(r) &= \pi r \\ g(r) &= -r^2. \end{aligned}$$

Therefore,

$$\begin{aligned} u(r,\theta) &= \pi r \sin\theta - r^2 \sin 2\theta \\ &= \pi r \sin\theta - r^2(2\sin\theta\cos\theta) \\ &= r\sin\theta(\pi - 2r\cos\theta). \end{aligned}$$

This solution can be written in Cartesian coordinates by writing $x = r\cos\theta$ and $y = r\sin\theta$.

$$u(x,y) = y(\pi - 2x)$$

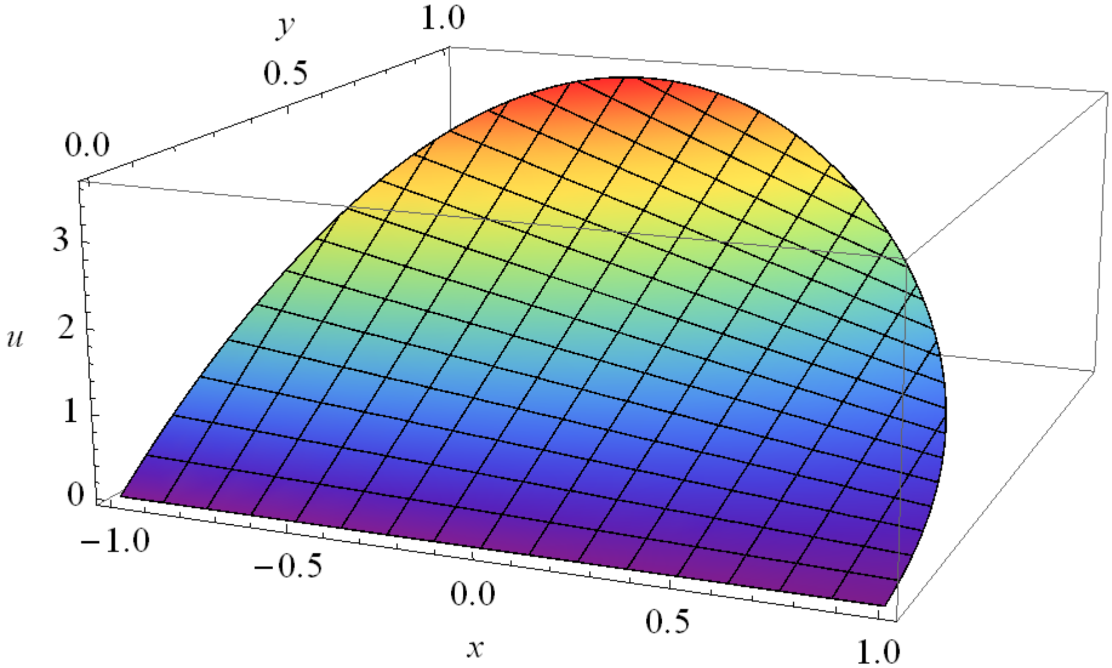

Figure 1: This is a plot of the two-dimensional solution surface $u(x, y)$ in three-dimensional xyu-space. Notice that the maximum and minimum values of u lie on the boundary (maximum principle).