Exercise 5

(a) Find the steady-state temperature distribution inside an annular plate \( \{1 < r < 2\} \), whose outer edge \((r = 2)\) is insulated, and on whose inner edge \((r = 1)\) the temperature is maintained as \(\sin^2 \theta\). (Find explicitly all the coefficients, etc.)

(b) Same, except \(u = 0\) on the outer edge.

Solution

The governing equation for the steady-state temperature \(u\) in a domain without heat sources is the Laplace equation.

\[ \nabla^2 u = 0 \]

Since the domain we want to solve it in is an annulus \((1 < r < 2)\), we choose to write the Laplacian operator in polar coordinates.

\[
\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0, \quad 1 < r < 2, \quad 0 < \theta < 2\pi \tag{1}
\]

Part (a)

The insulation at \(r = 2\) means that the derivative in the \(r\)-direction (normal to the circular boundary) is zero. The temperature at \(r = 1\) is given to be \(\sin^2 \theta\), which can be written as \(\frac{1}{2}(1 - \cos 2\theta)\).

\[
u(1, \theta) = \frac{1}{2} - \frac{1}{2} \cos 2\theta
\]
\[
u_r(2, \theta) = 0
\]

From the form of the inhomogeneous boundary condition we hypothesize that the solution has the form

\[ u(r, \theta) = \frac{1}{2} + g(r) \cos 2\theta. \]

Apply the boundary conditions for \(u\) to determine the boundary conditions for \(g\).

\[
u(1, \theta) = \frac{1}{2} + g(1) \cos 2\theta = \frac{1}{2} - \frac{1}{2} \cos 2\theta \quad \rightarrow \quad g(1) = -\frac{1}{2}
\]
\[
u_r(2, \theta) = g'(2) \cos 2\theta = 0 \quad \rightarrow \quad g'(2) = 0 \tag{2}
\]

In order to determine \(g(r)\), substitute the expression for \(u(r, \theta)\) into equation (1).

\[
\frac{\partial^2}{\partial r^2} \left[ \frac{1}{2} + g(r) \cos 2\theta \right] + \frac{1}{r} \frac{\partial}{\partial r} \left[ \frac{1}{2} + g(r) \cos 2\theta \right] + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \left[ \frac{1}{2} + g(r) \cos 2\theta \right] = 0
\]

Evaluate the derivatives.

\[
g''(r) \cos 2\theta + \frac{1}{r} g'(r) \cos 2\theta + \frac{1}{r^2} g(r)(-4 \cos 2\theta) = 0
\]

Multiply both sides by \(r^2/ \cos 2\theta\).

\[
r^2 g'' + rg' - 4g = 0
\]

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Since $\theta$ is not present in this equation, the hypothesis for $u(r, \theta)$ is legitimate. This is an equidimensional ODE for $g$, so it has solutions of the form

$$g(r) = r^m \quad \rightarrow \quad g'(r) = mr^{m-1} \quad \rightarrow \quad g''(r) = m(m - 1)r^{m-2}.$$  

Substitute these expressions into the ODE to determine the constants $m$.

$$m(m - 1)r^m + mr^m - 4r^m = 0$$

Divide both sides by $r^m$.

$$m(m - 1) + m - 4 = 0$$

Solve for $m$.

$$m^2 - 4 = 0 \quad \rightarrow \quad m = \{\pm 2\}$$

Consequently,

$$g(r) = C_1r^2 + C_2r^{-2}.$$  

Now apply the boundary conditions for $g$ to determine $C_1$ and $C_2$.

$$g(1) = C_1 + C_2 = -\frac{1}{2}$$

$$g'(2) = 4C_1 - \frac{C_2}{4} = 0$$

Solving the system of equations yields $C_1 = -1/34$ and $C_2 = -8/17$.

$$g(r) = -\frac{1}{34}r^2 - \frac{8}{17}r^{-2}$$

$$= -\frac{r^4 + 16}{34r^2}$$

Therefore,

$$u(r, \theta) = \frac{1}{2} - \frac{r^4 + 16}{34r^2} \cos 2\theta.$$  

This solution can be written in Cartesian coordinates by writing $\cos 2\theta = 2\cos^2 \theta - 1$ and then using $r^2 = x^2 + y^2$ and $\cos \theta = x/\sqrt{x^2 + y^2}$.

$$u(x, y) = \frac{1}{2} - \frac{(x^2 + y^2)^2 + 16}{34(x^2 + y^2)} \left( \frac{2x^2}{x^2 + y^2} - 1 \right)$$

$$= \frac{1}{2} - \frac{(x^2 + y^2)^2 + 16}{34(x^2 + y^2)^2} (x^2 - y^2)$$
Figure 1: This is a plot of the two-dimensional solution surface \( u(x,y) \) in three-dimensional \( xyu \)-space. Notice that the maximum and minimum values of \( u \) lie on the boundary (maximum principle).

**Part (b)**

Here the boundary conditions are

\[
\begin{align*}
  u(1, \theta) &= \frac{1}{2} - \frac{1}{2} \cos 2\theta \\
  u(2, \theta) &= 0
\end{align*}
\]

From the form of the inhomogeneous boundary condition we hypothesize that the solution has the form

\[
  u(r, \theta) = f(r) + h(r) \cos 2\theta.
\]

Apply the boundary conditions for \( u \) to determine the boundary conditions for \( f \) and \( h \).

\[
\begin{align*}
  u(1, \theta) &= f(1) + h(1) \cos 2\theta = \frac{1}{2} - \frac{1}{2} \cos 2\theta \quad \rightarrow \quad f(1) = \frac{1}{2} \quad \text{and} \quad h(1) = -\frac{1}{2} \\
  u(2, \theta) &= f(2) + h(2) \cos 2\theta = 0 \quad \rightarrow \quad f(2) = 0 \quad \text{and} \quad h(2) = 0
\end{align*}
\]

In order to determine \( f(r) \) and \( h(r) \), substitute the expression for \( u(r, \theta) \) into equation (1).

\[
\frac{\partial^2}{\partial r^2} [f(r) + h(r) \cos 2\theta] + \frac{1}{r} \frac{\partial}{\partial r} [f(r) + h(r) \cos 2\theta] + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} [f(r) + h(r) \cos 2\theta] = 0
\]

Evaluate the derivatives.

\[
f''(r) + h''(r) \cos 2\theta + \frac{1}{r} [f'(r) + h'(r) \cos 2\theta] + \frac{1}{r^2} [-4h(r) \cos 2\theta] = 0
\]

Expand the left side.

\[
f''(r) + \frac{1}{r} f'(r) + h''(r) \cos 2\theta + \frac{1}{r} h'(r) \cos 2\theta - \frac{4}{r^2} h(r) \cos 2\theta = 0
\]

If we set

\[
f''(r) + \frac{1}{r} f'(r) = 0, \quad (3)
\]
then the previous equation reduces to

\[ h''(r) \cos 2\theta + \frac{1}{r} h'(r) \cos 2\theta - \frac{4}{r^2} h(r) \cos 2\theta = 0. \]

Dividing both sides by \( \cos 2\theta \),

\[ h''(r) + \frac{1}{r} h'(r) - \frac{4}{r^2} h(r) = 0, \]  

we obtain an equation that is independent of \( \theta \), proving the legitimacy of the hypothesis.

Equation (3) is first-order in \( f' \), so we multiply both sides by the integrating factor \( I \).

\[ I = \exp \left( \int \frac{1}{s} \, ds \right) = \exp(\ln r) = r \]

Doing so gives us

\[ r f'' + f' = 0. \]

The left side can be written as \( d/dr(If') \) as a result of the product rule.

\[ \frac{d}{dr}(rf') = 0 \]

Integrate both sides with respect to \( r \).

\[ rf' = C_3 \]

Divide both sides by \( r \).

\[ f' = \frac{C_3}{r} \]

Integrate both sides with respect to \( r \) once more.

\[ f(r) = C_3 \ln r + C_4 \]

Apply the two boundary conditions for \( f \) to determine \( C_3 \) and \( C_4 \).

\[ f(1) = C_4 = \frac{1}{2} \]

\[ f(2) = C_3 \ln 2 + C_4 = 0 \quad \rightarrow \quad C_3 = -\frac{1}{2\ln 2} \]

Consequently,

\[ f(r) = -\frac{1}{2\ln 2} \ln r + \frac{1}{2} \]

\[ = \frac{1}{2} \left( 1 - \frac{\ln r}{\ln 2} \right). \]

Now equation (4) will be solved. Multiply both sides of it by \( r^2 \).

\[ r^2 h'' + rh' - 4h = 0 \]

This is identical to the ODE solved earlier for \( g \), so it has the same general solution.

\[ h(r) = C_5 r^2 + C_6 r^{-2} \]

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Apply the two boundary conditions for $h$ to determine $C_5$ and $C_6$.

\[ h(1) = C_5 + C_6 = -\frac{1}{2} \]
\[ h(2) = 4C_5 + \frac{C_6}{4} = 0 \]

Solving this system of equations yields $C_5 = 1/30$ and $C_6 = -8/15$. Consequently,

\[ h(r) = \frac{1}{30} r^2 - \frac{8}{15r^2} \]
\[ = \frac{r^4 - 16}{30r^2}. \]

Therefore,

\[ u(r, \theta) = \frac{1}{2} \left( 1 - \ln r^2 + \frac{r^4 - 16}{30r^2} \cos 2\theta. \right) \]

This solution can be written in Cartesian coordinates by writing $\cos 2\theta = 2 \cos^2 \theta - 1$ and then using $r^2 = x^2 + y^2$ and $\cos \theta = x/\sqrt{x^2 + y^2}$.

\[ u(x, y) = \frac{1}{2} \left( 1 - \ln \sqrt{x^2 + y^2} \right) + \frac{(x^2 + y^2)^2 - 16}{30(x^2 + y^2)} \left( \frac{2x^2}{x^2 + y^2} - 1 \right) \]
\[ = \frac{1}{2} \left( 1 - \ln \sqrt{x^2 + y^2} \right) + \frac{(x^2 + y^2)^2 - 16}{30(x^2 + y^2)^2} (x^2 - y^2). \]

Figure 2: This is a plot of the two-dimensional solution surface $u(x, y)$ in three-dimensional $xyu$-space. Notice that the maximum and minimum values of $u$ lie on the boundary (maximum principle).