Exercise 9

Solve \( u_{xx} + u_{yy} = 0 \) in the wedge \( r < a, \ 0 < \theta < \beta \) with the BCs

\[
\begin{align*}
    u = \theta & \quad \text{on} \ r = a, \quad u = 0 & \quad \text{on} \ \theta = 0, \quad \text{and} \quad u = \beta & \quad \text{on} \ \theta = \beta.
\end{align*}
\]

(Hint: Look for a function independent of \( r \).)

Solution

The domain is a wedge, so we choose to expand the Laplacian operator in polar coordinates. The boundary value problem to solve then is the following.

\[
\begin{align*}
    u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} &= 0, \quad r < a, \ 0 < \theta < \beta \\
    u(0, \theta) &= \text{bounded}, \quad u(r, 0) = 0 \\
    u(a, \theta) &= \theta, \quad u(r, \beta) = \beta
\end{align*}
\]

From the boundary conditions we assume that the solution is only a function of \( \theta \), \( u = u(\theta) \). As a result, the Laplace equation simplifies to an ODE,

\[
\frac{1}{r^2} \frac{d^2 u}{d\theta^2} = 0 \quad \Rightarrow \quad \frac{d^2 u}{d\theta^2} = 0,
\]

with boundary conditions,

\[
\begin{align*}
    u(0) &= 0 \\
    u(\beta) &= \beta.
\end{align*}
\]

Integrate both sides of the ODE with respect to \( \theta \).

\[
\frac{du}{d\theta} = C_1
\]

Integrate both sides of the ODE with respect to \( \theta \) once more.

\[
u(\theta) = C_1 \theta + C_2
\]

Apply the boundary conditions here to determine \( C_1 \) and \( C_2 \).

\[
\begin{align*}
    u(0) &= C_2 = 0 \\
    u(\beta) &= C_1 \beta + C_2 = \beta
\end{align*}
\]

Solving the second equation for \( C_1 \) gives \( C_1 = 1 \). Therefore,

\[
u(\theta) = \theta.
\]

In Cartesian coordinates this is

\[
u(x, y) = \tan^{-1} \frac{y}{x}.
\]
Figure 1: This is a plot of the two-dimensional solution surface $u(x, y)$ in three-dimensional $xyu$-space for $a = 1$ and $\beta = \pi/3$. Notice that the maximum and minimum values of $u$ lie on the boundary (maximum principle).