Exercise 3

Prove the uniqueness of the Robin problem $\partial u/\partial n + a(\mathbf{x})u(\mathbf{x}) = h(\mathbf{x})$ provided that $a(\mathbf{x}) > 0$ on the boundary.

Solution

The Robin problem is

$$\Delta u = f \quad \text{in } D$$
$$\frac{\partial u}{\partial n} + au = h \quad \text{on bdy } D.$$

Suppose that in addition to u there is a second solution v to this problem.

$$\Delta v = f \quad \text{in } D$$
$$\frac{\partial v}{\partial n} + av = h \quad \text{on bdy } D$$

Subtract the respective sides of the equations valid in D as well as the respective sides of the equations valid on bdy D.

$$\Delta u - \Delta v = f - f \quad \text{in } D$$
$$\frac{\partial u}{\partial n} + au - \frac{\partial v}{\partial n} - av = h - h \quad \text{on bdy } D$$

Factor the operator in the first equation.

$$\Delta(u - v) = 0 \quad \text{in } D$$
$$\frac{\partial}{\partial n}(u - v) + a(u - v) = 0 \quad \text{on bdy } D$$

Let $w = u - v$.

$$\Delta w = 0 \quad \text{in } D$$
$$\frac{\partial w}{\partial n} + aw = 0 \quad \text{on bdy } D$$

Taking the two arbitrary functions to be w, Green's first identity says that

$$\iint_{\text{bdy } D} w\frac{\partial w}{\partial n}\, dS = \iiint_D |\nabla w|^2\, dV + \iiint_D w\Delta w\, dV$$
$$\iint_{\text{bdy } D} w(-aw)\, dS = \iiint_D |\nabla w|^2\, dV$$
$$-\iint_{\text{bdy } D} aw^2\, dS = \iiint_D |\nabla w|^2\, dV.$$

Since a is positive, the integral on the left is positive. The integral on the right is also positive because its integrand is positive. Due to the minus sign, both sides must be equal to zero.

$$-\iint_{\text{bdy } D} aw^2\, dS = \iiint_D |\nabla w|^2\, dV = 0$$

This equation is equivalent to these two.

$$\begin{cases} \displaystyle\iint_{\text{bdy } D} aw^2\,dS = 0 \\ \displaystyle\iiint_D |\nabla w|^2\,dV = 0 \end{cases}$$

By the vanishing theorem, both integrands must be equal to zero.

$$\begin{cases} aw^2 = 0 & \text{on bdy } D \\ |\nabla w|^2 = 0 & \text{in } D \end{cases}$$

$$\begin{cases} w^2 = 0 & \text{on bdy } D \\ \nabla w = \mathbf{0} & \text{in } D \end{cases}$$

$$\begin{cases} w = 0 & \text{on bdy } D \\ w = \text{constant} & \text{in } D \end{cases}$$

In order for w to be consistent with its value on the boundary of D, this constant has to be zero.

$$\begin{cases} w = 0 & \text{on bdy } D \\ w = 0 & \text{in } D \end{cases}$$

$w = 0$ implies that $u = v$, which means that the two solutions to the Robin problem must be one and the same function.