Exercise 5

Prove Dirichlet's principle for the Neumann boundary condition. It asserts that among *all* real-valued functions $w(\mathbf{x})$ on D the quantity

$$E[w] = \frac{1}{2} \iiint_D |\nabla w|^2 \, d\mathbf{x} - \iint_{\text{bdy } D} hw \, dS$$

is the smallest for $w = u$, where u is the solution of the Neumann problem

$$-\Delta u = 0 \quad \text{in } D, \qquad \frac{\partial u}{\partial n} = h(\mathbf{x}) \quad \text{on bdy } D.$$

It is required to assume that the average of the given function $h(\mathbf{x})$ is zero (by Exercise 6.1.11).

Notice three features of this principle:

(i) There is *no constraint at all* on the trial functions $w(\mathbf{x})$.

(ii) The function $h(\mathbf{x})$ appears in the energy.

(iii) The functional $E[w]$ does not change if a constant is added to $w(\mathbf{x})$.

(*Hint:* Follow the method in Section 7.1.)