Exercise 7

(*Rayleigh-Ritz approximation* to the harmonic function u in D with $u = h$ on bdy D.) Let $w_0, w_1, \ldots, w_n$ be arbitrary functions such that $w_0 = h$ on bdy D and $w_1 = \cdots = w_n = 0$ on bdy D. The problem is to find constants $c_1, \ldots, c_n$ so that

$$w_0 + c_1 w_1 + \cdots + c_n w_n \quad \text{has the least possible energy.}$$

Show that the constants must solve the linear system

$$\sum_{k=1}^{n} (\nabla w_j, \nabla w_k)\, c_k = -(\nabla w_0, \nabla w_j) \quad \text{for } j = 1, 2, \ldots, n.$$

Solution

According to Dirichlet's principle, the function with the lowest energy in D that satisfies the boundary condition of D is harmonic. As a result, $w_0 + c_1 w_1 + \cdots + c_n w_n$ satisfies the Laplace equation.

$$\Delta(w_0 + c_1 w_1 + \cdots + c_n w_n) = 0$$

The Laplacian operator $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$ is linear.

$$\Delta w_0 + c_1 \Delta w_1 + \cdots + c_n \Delta w_n = 0$$

In order to determine the system of equations for the coefficients, multiply both sides by w_j, where j is an integer $1 \le j \le n$,

$$w_j \Delta w_0 + c_1 w_j \Delta w_1 + \cdots + c_n w_j \Delta w_n = 0$$

and then integrate both sides over the volume of D.

$$\iiint_D w_j \Delta w_0 \, dV + c_1 \iiint_D w_j \Delta w_1 \, dV + \cdots + c_n \iiint_D w_j \Delta w_n \, dV = 0 \tag{1}$$

Green's first identity states that for any two functions, u and v, defined in D,

$$\iint_{\text{bdy } D} v \frac{\partial u}{\partial n} \, dS = \iiint_D \nabla v \cdot \nabla u \, dV + \iiint_D v \Delta u \, dV.$$

Let $u = w_i$ and $v = w_j$, where i is an integer $0 \le i \le n$.

$$\iint_{\text{bdy } D} w_j \frac{\partial w_i}{\partial n} \, dS = \iiint_D \nabla w_j \cdot \nabla w_i \, dV + \iiint_D w_j \Delta w_i \, dV$$

Since $w_j = 0$ on the boundary of D, the surface integral on the left is zero.

$$0 = \iiint_D \nabla w_j \cdot \nabla w_i \, dV + \iiint_D w_j \Delta w_i \, dV$$

Consequently,

$$\begin{aligned}\iiint_D w_j \Delta w_i \, dV &= -\iiint_D \nabla w_j \cdot \nabla w_i \, dV \\ &= -(\nabla w_j, \nabla w_i),\end{aligned}$$

and equation (1) becomes

$$-(\nabla w_j, \nabla w_0) - c_1(\nabla w_j, \nabla w_1) - \cdots - c_n(\nabla w_j, \nabla w_n) = 0$$

$$-(\nabla w_j, \nabla w_0) - \sum_{k=1}^{n} c_k(\nabla w_j, \nabla w_k) = 0.$$

Therefore, the system of equations for the coefficients is

$$\sum_{k=1}^{n} (\nabla w_j, \nabla w_k)\, c_k = -\,(\nabla w_0, \nabla w_j) \quad \text{for } j = 1, 2, \ldots, n.$$