Exercise 2

Prove the uniqueness up to constants of the Neumann problem using the energy method.

Solution

The Neumann problem is

\[ \Delta u = f \quad \text{in } D \]
\[ \frac{\partial u}{\partial n} = h \quad \text{on } \partial D. \]

Suppose that in addition to \( u \) there is a second solution \( v \) to this problem.

\[ \Delta v = f \quad \text{in } D \]
\[ \frac{\partial v}{\partial n} = h \quad \text{on } \partial D \]

Subtract the respective sides of the equations valid in \( D \) as well as the respective sides of the equations valid on \( \partial D \).

\[ \Delta u - \Delta v = f - f \quad \text{in } D \]
\[ \frac{\partial u}{\partial n} - \frac{\partial v}{\partial n} = h - h \quad \text{on } \partial D \]

Factor the operator in the first equation.

\[ \Delta(u - v) = 0 \quad \text{in } D \]
\[ \frac{\partial}{\partial n}(u - v) = 0 \quad \text{on } \partial D \]

Let \( w = u - v \).

\[ \Delta w = 0 \quad \text{in } D \]
\[ \frac{\partial w}{\partial n} = 0 \quad \text{on } \partial D \]

Taking the two arbitrary functions to be \( w \), Green’s first identity says that

\[ \iiint_{\partial D} w \frac{\partial w}{\partial n} dS = \iiint_D |\nabla w|^2 dV + \iiint_D w \Delta w dV \]
\[ 0 = \iiint_D |\nabla w|^2 dV. \]

By the vanishing theorem, the integrand is zero.

\[ |\nabla w|^2 = 0 \]
\[ \nabla w = 0 \]
\[ w = \text{constant} \]

Therefore, \( u = v + \text{constant} \), which means the solution to the Neumann problem is unique up to a constant.

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