Exercise 5

Prove Dirichlet's principle for the Neumann boundary condition. It asserts that among all real-valued functions \( w(x) \) on \( D \) the quantity

\[
E[w] = \frac{1}{2} \int_{D} |\nabla w|^2 \, dx - \int_{\partial D} hw \, dS
\]

is the smallest for \( w = u \), where \( u \) is the solution of the Neumann problem

\[
-\Delta u = 0 \quad \text{in} \quad D, \quad \frac{\partial u}{\partial n} = h(x) \quad \text{on} \quad \partial D.
\]

It is required to assume that the average of the given function \( h(x) \) is zero (by Exercise 6.1.11).

Notice three features of this principle:

(i) There is no constraint at all on the trial functions \( w(x) \).

(ii) The function \( h(x) \) appears in the energy.

(iii) The functional \( E[w] \) does not change if a constant is added to \( w(x) \).

(Hint: Follow the method in Section 7.1.)

Solution

Let \( u \) satisfy the Neumann problem,

\[
\Delta u = 0 \quad \text{in} \quad D, \quad \frac{\partial u}{\partial n} = h \quad \text{on} \quad \partial D
\]

and let \( w \) be a real-valued function in \( D \) that satisfies \( w = u - v \). As a result, the energy becomes

\[
E[w] = \frac{1}{2} \int_{D} |\nabla w|^2 \, dv - \int_{\partial D} hw \, dS
\]

\[
= \frac{1}{2} \int_{D} \nabla u \cdot \nabla w \, dv - \int_{\partial D} hw \, dS
\]

\[
= \frac{1}{2} \int_{D} \nabla (u - v) \cdot \nabla (u - v) \, dv - \int_{\partial D} h(u - v) \, dS
\]

\[
= \frac{1}{2} \int_{D} (\nabla u - \nabla v) \cdot (\nabla u - \nabla v) \, dv - \int_{\partial D} (hu - hv) \, dS
\]

\[
= \frac{1}{2} \int_{D} (\nabla u \cdot \nabla u - 2\nabla u \cdot \nabla v + \nabla v \cdot \nabla v) \, dv - \int_{\partial D} hu \, dS + \int_{\partial D} hv \, dS
\]

\[
= \frac{1}{2} \int_{D} (|\nabla u|^2 - 2\nabla u \cdot \nabla v + |\nabla v|^2) \, dv - \int_{\partial D} hu \, dS + \int_{\partial D} hv \, dS.
\]
Split up the integrals and then use the definition of energy to combine the first and fourth terms.

\[
E[w] = \frac{1}{2} \iiint_D |\nabla u|^2 \, dV + \frac{1}{2} \iiint_D |\nabla v|^2 \, dV - \iiint_D \nabla u \cdot \nabla v \, dV - \int_{\partial D} hu \, dS + \int_{\partial D} hv \, dS
\]

\[
= E[u] + \frac{1}{2} \iiint_D |\nabla v|^2 \, dV - \iiint_D \nabla u \cdot \nabla v \, dV + \int_{\partial D} hv \, dS
\]

Green’s first identity says that for two arbitrary functions, \(u\) and \(v\),

\[
\int_{\partial D} v \frac{\partial u}{\partial n} \, dS = \iiint_D \nabla u \cdot \nabla v \, dV + \iiint_D v \Delta u \, dV \rightarrow \iiint_D \nabla u \cdot \nabla v \, dV = \int_{\partial D} v \frac{\partial u}{\partial n} \, dS - \iiint_D v \Delta u \, dV
\]

so the previous equation becomes

\[
E[w] = E[u] + \frac{1}{2} \iiint_D |\nabla v|^2 \, dV - \int_{\partial D} v(h) \, dS + \iiint_D v(0) \, dV + \int_{\partial D} hv \, dS
\]

\[
= E[u] + \frac{1}{2} \iiint_D |\nabla v|^2 \, dV - \int_{\partial D} hv \, dS + \iiint_D hv \, dS
\]

\[
= E[u] + \frac{1}{2} \iiint_D |\nabla v|^2 \, dV.
\]

The second term is positive, so \(E[w] \geq E[u]\). Therefore, the energy \(E\) is minimized if \(w = u\).