Exercise 6

Let \( A \) and \( B \) be two disjoint bounded spatial domains, and let \( D \) be their exterior. So \( \text{bdy} \, D = \text{bdy} \, A \cup \text{bdy} \, B \). Consider a harmonic function \( u(x) \) in \( D \) that tends to zero at infinity, which is constant on \( \text{bdy} \, A \) and \( \text{constant} \) on \( \text{bdy} \, B \), and which satisfies

\[
\int_{\text{bdy} \, A} \frac{\partial u}{\partial n} \, dS = Q > 0 \quad \text{and} \quad \int_{\text{bdy} \, B} \frac{\partial u}{\partial n} \, dS = 0.
\]

[Interpretation: The harmonic function \( u(x) \) is the electrostatic potential of two conductors, \( A \) and \( B \); \( Q \) is the charge on \( A \), while \( B \) is uncharged.]

(a) Show that the solution is unique. \( \text{(Hint: Use the Hopf maximum principle.)} \)

(b) Show that \( u \geq 0 \) in \( D \). \( \text{(Hint: If not, then } u(x) \text{ has a negative minimum. Use the Hopf principle again.)} \)

(c) Show that \( u > 0 \) in \( D \).

Solution

A harmonic function is a function that satisfies the Laplace equation. Suppose that there are two solutions, \( u = u(x, y, z) \) and \( v = (x, y, z) \), for the potential.

\[
\Delta u = 0 \quad \text{in } D \quad \int_{\text{bdy} \, A} \frac{\partial u}{\partial n} \, dS = Q \quad \int_{\text{bdy} \, B} \frac{\partial u}{\partial n} \, dS = 0 \quad u = \begin{cases} C_1 & \text{on } \text{bdy } A \\ C_2 & \text{on } \text{bdy } B \end{cases} \quad \lim_{|x| \to \infty} u = 0
\]

\[
\Delta v = 0 \quad \text{in } D \quad \int_{\text{bdy} \, A} \frac{\partial v}{\partial n} \, dS = Q \quad \int_{\text{bdy} \, B} \frac{\partial v}{\partial n} \, dS = 0 \quad v = \begin{cases} C_3 & \text{on } \text{bdy } A \\ C_4 & \text{on } \text{bdy } B \end{cases} \quad \lim_{|x| \to \infty} v = 0
\]

Subtract both sides of each equation on the bottom from those of the equation above it.

\[
\Delta (u - v) = 0 \quad \int_{\text{bdy} \, A} \frac{\partial}{\partial n} (u - v) \, dS = Q - Q = 0 = \int_{\text{bdy} \, B} \frac{\partial}{\partial n} (u - v) \, dS = 0
\]

\[
u - v = \begin{cases} C_1 - C_3 & \text{on } \text{bdy } A \\ C_2 - C_4 & \text{on } \text{bdy } B \end{cases} \quad \lim_{|x| \to \infty} (u - v) = 0
\]

Let \( w \) be the difference of \( u \) and \( v \): \( w = u - v \).

\[
\Delta w = 0 \quad \int_{\text{bdy} \, A} \frac{\partial w}{\partial n} \, dS = 0 = \int_{\text{bdy} \, B} \frac{\partial w}{\partial n} \, dS = 0
\]

\[
w = \begin{cases} C_1 - C_3 & \text{on } \text{bdy } A \\ C_2 - C_4 & \text{on } \text{bdy } B \end{cases} \quad \lim_{|x| \to \infty} w = 0
\]

Setting both functions equal to \( w \) in Green’s first identity gives

\[
\int_{\text{bdy} \, D} w \frac{\partial w}{\partial n} \, dS = \int_{D} \nabla w \cdot \nabla w \, dV + \int_{D} w \Delta w \, dV.
\]
Since $\Delta w = 0$ in $D$, the second term on the right side is zero.

$$\int_{\partial D} w \frac{\partial w}{\partial n} dS = \int_D \nabla w \cdot \nabla w \, dV$$

Use the fact that $\partial D = \partial A \cup \partial B$.

$$\int_{\partial A \cup \partial B} w \frac{\partial w}{\partial n} dS = \int_D \nabla w \cdot \nabla w \, dV$$

Split up the integral on the left side.

$$\int_{\partial A} w \frac{\partial w}{\partial n} dS + \int_{\partial B} w \frac{\partial w}{\partial n} dS = \int_D \nabla w \cdot \nabla w \, dV$$

Substitute the value of $w$ on each boundary.

$$\int_{\partial A} (C_1 - C_3) \frac{\partial w}{\partial n} dS + \int_{\partial B} (C_2 - C_4) \frac{\partial w}{\partial n} dS = \int_D \nabla w \cdot \nabla w \, dV$$

Bring the constants in front of the integrals.

$$\left( C_1 - C_3 \right) \int_{\partial A} \frac{\partial w}{\partial n} dS + \left( C_2 - C_4 \right) \int_{\partial B} \frac{\partial w}{\partial n} dS$$

As a result,

$$0 = \int_D \nabla w \cdot \nabla w \, dV$$

$$= \int_D \langle w_x, w_y, w_z \rangle \cdot \langle w_x, w_y, w_z \rangle \, dV$$

$$= \int_D (w_x^2 + w_y^2 + w_z^2) \, dV.$$

By the vanishing theorem, the integrand must be zero.

$$w_x^2 + w_y^2 + w_z^2 = 0$$

This implies that $w_x$, $w_y$, and $w_z$ are zero individually,

$$w_x = 0$$
$$w_y = 0$$
$$w_z = 0,$$

which means that $w$ is a constant in $D$. In order to satisfy the condition at infinity, this constant must be zero.

$$w = 0$$

Therefore, the solution for the potential is unique.

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Setting both functions equal to $u$ in Green’s first identity gives

$$\int_{\partial D} u \frac{\partial u}{\partial n} dS = \int_D \nabla u \cdot \nabla u dV + \int_D u \Delta u dV.$$ 

Since $\Delta u = 0$ in $D$, the second term on the right side is zero.

$$\int_{\partial D} u \frac{\partial u}{\partial n} dS = \int_D \nabla u \cdot \nabla u dV$$

Use the fact that $\partial D = \partial A \cup \partial B$.

$$\int_{\partial A \cup \partial B} u \frac{\partial u}{\partial n} dS = \int_D \nabla u \cdot \nabla u dV$$

Split up the integral on the left side.

$$\int_{\partial A} u \frac{\partial u}{\partial n} dS + \int_{\partial B} u \frac{\partial u}{\partial n} dS = \int_D \nabla u \cdot \nabla u dV$$

Substitute the value of $u$ on each boundary.

$$\int_{\partial A} (C_1) \frac{\partial u}{\partial n} dS + \int_{\partial B} (C_2) \frac{\partial u}{\partial n} dS = \int_D \nabla u \cdot \nabla u dV$$

Bring the constants in front of the integrals.

$$C_1 \int_{\partial A} \frac{\partial u}{\partial n} dS + C_2 \int_{\partial B} \frac{\partial u}{\partial n} dS = \int_D \nabla u \cdot \nabla u dV$$

$$C_1 Q = \int_D (u_x^2 + u_y^2 + u_z^2) dV$$

The potential at the boundary of $A$ is now known.

$$C_1 = \frac{1}{Q} \int_D (u_x^2 + u_y^2 + u_z^2) dV$$

It’s positive because both $Q$ and the integrand are positive. The fact that conductor $B$ is uncharged (or grounded) means that the potential at its surface is zero: $C_2 = 0$.

$$u = \begin{cases} \frac{1}{Q} \int_D (u_x^2 + u_y^2 + u_z^2) dV & \text{on } \partial A \\ 0 & \text{on } \partial B \end{cases}$$

According to the maximum principle for the Laplace equation, the maximum and minimum of $u$ can only be attained on $\partial A$ or $\partial B$. Therefore,

$$0 < u < \frac{1}{Q} \int_D (u_x^2 + u_y^2 + u_z^2) dV \quad \text{in } D.$$