Exercise 8

Consider the problem $u_{xx} + u_{yy} = 0$ in the triangle $\{x > 0,\ y > 0,\ 3x + y < 3\}$ with the boundary conditions

$$u(x,0) = 0 \qquad u(0,y) = y(3-y) \qquad u(x, 3-3x) = 0$$

Choose $w_0 = y(3-3x-y)$ and $w_1 = xy(3-3x-y)$. Find the Rayleigh–Ritz approximation $w_0 + c_1 w_1$ to u. That is, use Exercise 7 to find the constant c_1.

Solution

The quantity $w_0 + c_1 w_1$ satisfies the Laplace equation.

$$\Delta(w_0 + c_1 w_1) = 0$$

The Laplacian operator $\Delta = \partial_x^2 + \partial_y^2$ is linear.

$$\Delta w_0 + c_1 \Delta w_1 = 0$$

Multiply both sides by w_1.

$$w_1 \Delta w_0 + c_1 w_1 \Delta w_1 = 0$$

Integrate both sides over the area D of the triangle in the xy-plane.

$$\iint_D w_1 \Delta w_0 \, dA + c_1 \iint_D w_1 \Delta w_1 \, dA = 0 \tag{1}$$

The analog of Green's first identity in two dimensions is

$$\int_{\text{bdy } D} v \frac{\partial u}{\partial n} \, ds = \iint_D \nabla v \cdot \nabla u \, dA + \iiint_D v \Delta u \, dA.$$

Let $u = w_i$ and $v = w_1$, where i is an integer $0 \le i \le 1$.

$$\int_{\text{bdy } D} w_1 \frac{\partial w_i}{\partial n} \, ds = \iint_D \nabla w_1 \cdot \nabla w_i \, dA + \iint_D w_1 \Delta w_i \, dA$$

Since $w_1 = 0$ on the boundary of D, the line integral on the left is zero.

$$0 = \iint_D \nabla w_1 \cdot \nabla w_i \, dA + \iint_D w_1 \Delta w_i \, dA$$

Consequently,

$$\iint_D w_1 \Delta w_i \, dA = -\iint_D \nabla w_1 \cdot \nabla w_i \, dA,$$

and equation (1) becomes

$$-\iint_D \nabla w_1 \cdot \nabla w_0 \, dA - c_1 \iint_D \nabla w_1 \cdot \nabla w_1 \, dA = 0.$$

Solve it for c_1 and evaluate the double integrals.

$$\begin{aligned}
c_1 &= -\frac{\displaystyle\iint_D \nabla w_1 \cdot \nabla w_0 \, dA}{\displaystyle\iint_D \nabla w_1 \cdot \nabla w_1 \, dA} \\
&= -\frac{\displaystyle\int_0^1 \int_0^{3-3x} \langle -y(-3+6x+y), x(3-3x-2y)\rangle \cdot \langle -3y, 3-3x-2y\rangle \, dy \, dx}{\displaystyle\int_0^1 \int_0^{3-3x} \langle -y(-3+6x+y), x(3-3x-2y)\rangle \cdot \langle -y(-3+6x+y), x(3-3x-2y)\rangle \, dy \, dx} \\
&= -\frac{\displaystyle\int_0^1 \int_0^{3-3x} [3y^2(-3+6x+y) + x(3-3x-2y)^2] \, dy \, dx}{\displaystyle\int_0^1 \int_0^{3-3x} [y^2(-3+6x+y)^2 + x^2(3-3x-2y)^2] \, dy \, dx} \\
&= -\frac{0.45}{1.5} \\
&= -0.3
\end{aligned}$$

Therefore, the Rayleigh–Ritz approximation $w_0 + c_1 w_1$ to u is

$$u \approx y(3-3x-y) - 0.3xy(3-3x-y).$$