Exercise 3

Give yet another derivation of the mean value property in three dimensions by choosing D to be a ball and $\mathbf{x}_0$ its center in the representation formula (1).

Solution

Suppose that $\Delta u = 0$ in a domain D of space. Then u can be expressed as an integral over the boundary of D by the representation formula in equation (1) of the text.

$$u(\mathbf{x}_0) = \iint_{\text{bdy } D} \left[-u(\mathbf{x})\frac{\partial}{\partial n}\left(\frac{1}{|\mathbf{x}-\mathbf{x}_0|}\right) + \frac{1}{|\mathbf{x}-\mathbf{x}_0|}\frac{\partial u}{\partial n}\right]\frac{dS}{4\pi} \tag{1}$$

We choose D to be a solid ball of radius R centered at $\mathbf{x}_0$.

$$u(\mathbf{x}_0) = \iint_{\substack{(x-x_0)^2+(y-y_0)^2\\+(z-z_0)^2=R^2}} \left[-u(\mathbf{x})\frac{\partial}{\partial n}\left(\frac{1}{|\mathbf{x}-\mathbf{x}_0|}\right) + \frac{1}{|\mathbf{x}-\mathbf{x}_0|}\frac{\partial u}{\partial n}\right]\frac{dS}{4\pi}$$

If we let $r = |\mathbf{x}-\mathbf{x}_0| = \sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}$, then the boundary can be expressed compactly as $r = R$.

$$u(\mathbf{x}_0) = \iint_{r=R} \left[-u(\mathbf{x})\frac{\partial}{\partial n}\left(\frac{1}{r}\right) + \frac{1}{r}\frac{\partial u}{\partial n}\right]\frac{dS}{4\pi}$$

The normal derivative to this ball is a radial derivative that points away from its center: $\partial/\partial n = \partial/\partial r$.

$$\begin{aligned}
u(\mathbf{x}_0) &= \iint_{r=R} \left[-u(\mathbf{x})\frac{\partial}{\partial r}\left(\frac{1}{r}\right) + \frac{1}{r}\frac{\partial u}{\partial r}\right]\frac{dS}{4\pi} \\
&= \iint_{r=R} \left[u(\mathbf{x})\left(\frac{1}{r^2}\right) + \frac{1}{r}\frac{\partial u}{\partial r}\right]\frac{dS}{4\pi} \\
&= \iint_{r=R} \left[u(\mathbf{x})\left(\frac{1}{R^2}\right) + \frac{1}{R}\frac{\partial u}{\partial r}\right]\frac{dS}{4\pi} \\
&= \frac{1}{4\pi R^2}\iint_{r=R} u(\mathbf{x})\,dS + \frac{1}{4\pi R}\iint_{r=R}\frac{\partial u}{\partial r}\,dS \\
&= \frac{\iint_{r=R} u(\mathbf{x})\,dS}{\iint_{r=R} dS} + \frac{1}{4\pi R}\iint_{r=R} \nabla u\cdot\hat{\mathbf{r}}\,dS \\
&= \bar{u} + \frac{1}{4\pi R}\iiint_{r\le R}\nabla\cdot\nabla u\,dV
\end{aligned}$$

The overbar here represents the average of the quantity below it over the sphere of radius R centered at $\mathbf{x}_0$. Since $\Delta u = \nabla\cdot\nabla u = 0$ in D, the second term is zero. Therefore, the value of u at the center of D is equal to the average of u over the boundary of D.