Exercise 1

Show that the Green's function is unique. (*Hint:* Take the difference of two of them.)

Solution

The Green's function $G_1 = G_1(\mathbf{x}; \mathbf{x}_0)$ satisfies the boundary value problem,

$$\begin{aligned} \Delta G_1 &= \delta(\mathbf{x} - \mathbf{x}_0) && \text{in } D \\ G_1 &= 0 && \text{on bdy } D, \end{aligned}$$

where δ is the Dirac delta function. Note that $\delta(\mathbf{x} - \mathbf{x}_0)$ is shorthand for
$\delta(x - x_0, y - y_0, z - z_0) = \delta(x - x_0)\delta(y - y_0)\delta(z - z_0)$. Suppose there is a second solution G_2 to this problem.

$$\begin{aligned} \Delta G_2 &= \delta(\mathbf{x} - \mathbf{x}_0) && \text{in } D \\ G_2 &= 0 && \text{on bdy } D \end{aligned}$$

Subtract the respective sides of each equation.

$$\begin{aligned} \Delta G_1 - \Delta G_2 &= \delta(\mathbf{x} - \mathbf{x}_0) - \delta(\mathbf{x} - \mathbf{x}_0) && \text{in } D \\ G_1 - G_2 &= 0 && \text{on bdy } D \end{aligned}$$

$$\begin{aligned} \Delta(G_1 - G_2) &= 0 && \text{in } D \\ G_1 - G_2 &= 0 && \text{on bdy } D \end{aligned}$$

Let $w = G_1 - G_2$.

$$\begin{aligned} \Delta w &= 0 && \text{in } D \\ w &= 0 && \text{on bdy } D \end{aligned}$$

Multiply both sides of the first equation by w.

$$w\Delta w = 0$$

Integrate both sides over the volume of D.

$$\iiint_D w\Delta w \, dV = 0 \tag{1}$$

Taking the two arbitrary functions to be w, Green's first identity says that

$$\iint_{\text{bdy } D} w \frac{\partial w}{\partial n} \, dS = \iiint_D |\nabla w|^2 \, dV + \iiint_D w\Delta w \, dV.$$

As a result, equation (1) becomes

$$\iint_{\text{bdy } D} w \frac{\partial w}{\partial n} \, dS - \iiint_D |\nabla w|^2 \, dV = 0.$$

Since w is zero on the boundary, the surface integral over the boundary of D is zero.

$$-\iiint_D |\nabla w|^2 \, dV = 0$$

Multiply both sides by -1.

$$\iiint_D |\nabla w|^2 \, dV = 0$$

By the vanishing theorem, the integrand is equal to zero.

$$|\nabla w|^2 = 0$$

$$\nabla w = \mathbf{0}$$

$$w = \text{constant}$$

In order to be consistent with the boundary condition, this constant has to be zero.

$$w = 0$$

This implies that $G_1 = G_2$, so the two Green's functions are one and the same function.