Exercise 1

Find the one-dimensional Green's function for the interval \((0, l)\). The three properties defining it can be restated as follows.

(i) It solves \(G''(x) = 0\) for \(x \neq x_0\) ("harmonic").

(ii) \(G(0) = G(l) = 0\).

(iii) \(G(x)\) is continuous at \(x_0\) and \(G(x) + \frac{1}{2}|x - x_0|\) is harmonic at \(x_0\).

Solution

The Green's function \(G = G(x; x_0)\) for the operator \(-\Delta\) satisfies the following boundary value problem.

\[
-\Delta G = \delta(x - x_0) \quad \text{in } D
\]

\[
G = 0 \quad \text{on } \partial D
\]

In one dimension on the interval \((0, l)\) this problem becomes

\[
-\frac{d^2 G}{dx^2} = \delta(x - x_0) \quad \text{in } (0, l)
\]

\[
G = 0 \quad \text{at } x = 0 \text{ and } x = l.
\]

If \(x \neq x_0\), then \(-d^2 G/dx^2 = 0\), and the general solution is obtained by integrating both sides with respect to \(x\) twice. To satisfy the boundary conditions, there is one solution to the left of \(x = x_0\) and one to the right of it.

\[
G(x; x_0) = \begin{cases} 
C_1 x + C_2 & \text{if } x < x_0 \\
C_3 x + C_4 & \text{if } x > x_0 
\end{cases}
\]

Four boundary conditions are necessary to determine \(C_1, C_2, C_3,\) and \(C_4\). The solution to the left of \(x = x_0\) must satisfy \(G = 0\) at \(x = 0\), and the solution to the right of \(x = x_0\) must satisfy \(G = 0\) at \(x = l\).

\[
C_1(0) + C_2 = 0 \quad C_3(l) + C_4 = 0 \\
C_2 = 0 \quad C_4 = -C_3 l
\]

For the third condition, we require that the Green's function be continuous at \(x = x_0\).

\[
C_1 x_0 + C_2 = C_3 x_0 + C_4 \]

\[
C_1 x_0 = C_3 x_0 - C_3 l
\]

\[
C_1 x_0 = C_3(x_0 - l) \quad (1)
\]

The fourth condition is obtained by integrating both sides of the differential equation over the spike located at \(x = x_0\).

\[
\int_{x_0-\epsilon}^{x_0+\epsilon} -\frac{d^2 G}{dx^2} \, dx = \int_{x_0-\epsilon}^{x_0+\epsilon} \delta(x - x_0) \, dx
\]

\[
-\frac{dG}{dx} \bigg|_{x_0+\epsilon}^{x_0-\epsilon} + \frac{dG}{dx} \bigg|_{x_0-\epsilon}^{x_0+\epsilon} = 1
\]

\[
-(C_3) + (C_1) = 1 \quad \rightarrow \quad C_1 = 1 + C_3
\]

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Substitute this result for $C_1$ into equation (1) and solve for $C_3$.

$$(1 + C_3)x_0 = C_3(x_0 - l) \quad \rightarrow \quad C_3 = \frac{-x_0}{l}$$

So then

$$C_1 = 1 - \frac{x_0}{l} = \frac{l - x_0}{l} \quad \text{and} \quad C_4 = -C_3l = x_0.$$ 

Plug in the constants to the formula for the Green's function.

$$G(x; x_0) = \begin{cases} \frac{l - x_0}{l} x & \text{if } x < x_0 \\ -\frac{x_0}{l} x + x_0 & \text{if } x > x_0 \end{cases}$$

Therefore,

$$G(x; x_0) = \begin{cases} \frac{l - x_0}{l} x & \text{if } 0 \leq x < x_0 \\ \frac{l - x}{l} x_0 & \text{if } x_0 < x \leq l \end{cases}.$$