Exercise 5

Notice that the function xy is harmonic in the half-plane $\{y > 0\}$ and vanishes on the boundary line $\{y = 0\}$. The function 0 has the same properties. Does this mean that the solution is not unique? Explain.

Solution

A harmonic function is a function that satisfies the Laplace equation. Since both 0 and xy satisfy the boundary value problem,

$$\begin{aligned} &\Delta u = 0, \qquad -\infty < x < \infty,\ y > 0 \\ &u(x, 0) = 0, \end{aligned}$$

its solution is not unique. In fact, there are infinitely many solutions besides 0 and xy, notably

$$\begin{aligned} u(x, y) &= \frac{1}{n^2} \sin nx \sinh ny \\ u(x, y) &= \frac{1}{n^2} \cos nx \sinh ny \\ u(x, y) &= \frac{1}{n^2} \cosh nx \sin ny. \end{aligned}$$

Uniqueness isn't the only issue with this boundary value problem. All of the solutions besides $u = 0$ lack stability. For example, at a very large value of x and at $y = 0$, $u = xy$ is zero; if y becomes just slightly positive, then $u = xy$ blows up. In other words, a tiny change in the data (x, y) does not result in a tiny change in u. The same is true for the solutions involving hyperbolic cosine. Those with hyperbolic sine are also unstable, as they diverge as $y \to \infty$.

A harmonic function can be interpreted physically as the steady-state temperature in a domain. For a half-plane that is subject to zero temperature along its boundary, the entire half-plane is expected to eventually have zero temperature regardless of its initial temperature profile. Physical intuition says there is a unique solution $u = 0$, and that's because there is an underlying assumption that the temperature is finite as $y \to \infty$. It would seem then that the boundary value problem can be made well-posed with the additional boundary condition,

$$\lim_{y\to\infty} u(x, y) \text{ exists},$$

but this is actually not necessary. Consider Green's first identity, which holds for any two functions, u and v, over any domain D and its boundary bdy D.

$$\int_{\text{bdy } D} v \frac{\partial u}{\partial n}\, ds = \iint_D \nabla v \cdot \nabla u \, dS + \iint_D v \Delta u \, dS$$

Let $v = u$.

$$\int_{\text{bdy } D} u \frac{\partial u}{\partial n}\, ds = \iint_D \nabla u \cdot \nabla u \, dS + \iint_D u \Delta u \, dS$$

If u satisfies the Laplace equation $\Delta u = 0$ and the boundary condition $u = 0$, then the term on the left side and the second term on the right side are zero.

$$0 = \iint_D \nabla u \cdot \nabla u \, dS$$

Evaluate the dot product.

$$\iint_D \langle u_x, u_y \rangle \cdot \langle u_x, u_y \rangle \, dS = 0$$

$$\iint_D (u_x^2 + u_y^2) \, dS = 0$$

Because the integrand is never negative, it must be zero (vanishing theorem).

$$u_x^2 + u_y^2 = 0$$

Since both u_x^2 and u_y^2 are never negative either, they must be zero.

$$u_x = 0$$
$$u_y = 0$$

That means u must be a constant in D. In order to satisfy the boundary condition, this constant must be zero.

$$u = 0$$

To conclude, even though there is an infinite number of solutions to the boundary value problem, only one of them satisfies Green's first identity. This also happens to be the physically relevant one.