Exercise 6

(a) Find the Green’s function for the half-plane \( \{(x, y) : y > 0\} \).

(b) Use it to solve the Dirichlet problem in the half-plane with boundary values \( h(x) \).

(c) Calculate the solution with \( u(x, 0) = 1 \).

Solution

The aim is to solve Poisson’s equation in the upper half plane that is subject to a boundary condition along \( y = 0 \).

\[
\Delta u = f(x, y), \quad -\infty < x < \infty, \quad y > 0 \\
u(x, 0) = h(x)
\]

A Green’s function representation for the solution can be obtained from Green’s second identity in two dimensions,

\[
\iint_D (u\Delta v - v\Delta u) \, dA = \int_{\partial D} \left( \frac{\partial v}{\partial n} - \frac{\partial u}{\partial n} \right) \, ds,
\]

which holds for any two functions, \( u \) and \( v \), over any domain and its boundary. Let \( v \) be the Green’s function: \( v = G \).

\[
\iint_D (u\Delta G - G\Delta u) \, dA = \int_{\partial D} \left( \frac{\partial G}{\partial n} - G \frac{\partial u}{\partial n} \right) \, ds \tag{1}
\]

If we require \( G = G(x, y; x_0, y_0) \) to satisfy

\[
\Delta G = \delta(x - x_0)\delta(y - y_0), \quad -\infty < x < \infty, \quad y > 0 \\
G = 0 \text{ at } y = 0,
\]

where \( (x_0, y_0) \) is a point in the half-plane, then equation (1) becomes

\[
\iint_D [u(x, y)\delta(x - x_0)\delta(y - y_0) - G(x, y; x_0, y_0)f(x, y)] \, dA = \int_{\partial D} \left( \frac{\partial G}{\partial n} - (0) \frac{\partial u}{\partial n} \right) \, ds.
\]

Since the domain is the upper half-plane, the outward unit normal vector is \( \hat{n} = -\hat{y} \), which means the normal derivative is \( \partial / \partial n = -\partial / \partial y \).

\[
\int_0^\infty \int_{-\infty}^\infty [u(x, y)\delta(x - x_0)\delta(y - y_0) - G(x, y; x_0, y_0)f(x, y)] \, dx \, dy = \int_0^\infty u(x, 0) \left( -\frac{\partial G}{\partial y} \right) \bigg|_{y=0} \, dx
\]

\[
u(x_0, y_0) - \int_0^\infty \int_{-\infty}^\infty G(x, y; x_0, y_0)f(x, y) \, dx \, dy = \int_{-\infty}^\infty h(x) \left( -\frac{\partial G}{\partial y} \right) \bigg|_{y=0} \, dx
\]

\[
u(x_0, y_0) = \int_0^\infty \int_{-\infty}^\infty G(x, y; x_0, y_0)f(x, y) \, dx \, dy - \int_{-\infty}^\infty h(x) \frac{\partial G}{\partial y} \bigg|_{y=0} \, dx
\]

Switch the roles of \( x_0 \) and \( y_0 \) with those of \( x \) and \( y \), respectively.

\[
u(x, y) = \int_0^\infty \int_{-\infty}^\infty G(x_0, y_0; x, y)f(x_0, y_0) \, dx_0 \, dy_0 - \int_{-\infty}^\infty h(x_0) \frac{\partial G}{\partial y_0} \bigg|_{y_0=0} \, dx_0
\]
Therefore, using the fact that the Green’s function is symmetric,

\[ u(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(x, y; x_0, y_0) f(x_0, y_0) \, dx_0 \, dy_0 - \int_{-\infty}^{\infty} h(x_0) \frac{\partial G}{\partial y_0} \bigg|_{y_0=0} \, dx_0. \]

The solution for Poisson’s equation is known, then, if the Green’s function in the upper half-plane can be determined. Begin by finding the Green’s function in the whole plane (no boundaries).

\[ \Delta g = \delta(x-x_0)\delta(y-y_0), \quad -\infty < x < \infty, \quad -\infty < y < \infty \]

g can be interpreted as the electrostatic potential, and \( \delta(x-x_0)\delta(y-y_0) \) can be interpreted as the charge density for a unit charge located at \((x_0, y_0)\). Since there are no boundaries, \( g \) is expected to vary solely as a function of the radial distance from \((x_0, y_0)\):

\[ g(r) = g(\sqrt{(x-x_0)^2 + (y-y_0)^2}) \]

Integrate both sides over a disk centered at \((x_0, y_0)\) with radius \( r \).

\[ \int_{(x-x_0)^2+(y-y_0)^2 \leq r^2} \Delta g \, dA = \int_{(x-x_0)^2+(y-y_0)^2 \leq r^2} \delta(x-x_0)\delta(y-y_0) \, dA \]

Since the disk contains \((x_0, y_0)\), the right side is 1. Write the Laplacian operator as \( \Delta = \nabla^2 \)

\[ \int_{(x-x_0)^2+(y-y_0)^2 \leq r^2} \nabla^2 g \, dA = 1 \]

\[ \int_{(x-x_0)^2+(y-y_0)^2 \leq r^2} \nabla \cdot \nabla g \, dA = 1 \]

and apply the two-dimensional divergence theorem.

\[ \int_{(x-x_0)^2+(y-y_0)^2 = r^2} \nabla g \cdot \hat{\mathbf{n}} \, ds = 1 \]

Here \( \hat{\mathbf{n}} \) is the unit vector normal to this disk at every point on the boundary.

\[ \int_{(x-x_0)^2+(y-y_0)^2 = r^2} \frac{dg}{d\varphi} \, ds = 1 \]

Because \( g \) only depends on \( \varphi \), its derivative is constant on the disk’s boundary.

\[ \frac{dg}{d\varphi} \int_{(x-x_0)^2+(y-y_0)^2 = r^2} ds = 1 \]

This line integral is just the disk’s circumference.

\[ \frac{dg}{d\varphi} (2\pi \varphi) = 1 \]

Divide both sides by \( 2\pi \varphi \).

\[ \frac{dg}{d\varphi} = \frac{1}{2\pi \varphi} \]

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Integrate both sides with respect to $r$.

$$g(r) = \frac{1}{2\pi} \ln r$$

The Green’s function for the whole plane is then

$$g(x, y; x_0, y_0) = \frac{1}{2\pi} \ln \sqrt{(x - x_0)^2 + (y - y_0)^2}.$$ 

Now that it’s known, the Green’s function for the upper half-plane can be determined by the method of images. A convocation of point charges in the whole plane will be arranged so that the boundary condition, $G = 0$ along $y = 0$, is satisfied.

Since the two charges have the same magnitude but opposite polarity and are equally spaced from every point along the $x$-axis, the potential at every point on the $x$-axis is zero. The half-plane Green’s function can now be written.

$$G(x, y; x_0, y_0) = +g(x, y; x_0, y_0) - g(x, y; x_0, -y_0), \quad y > 0$$

Since $g$ is defined over the whole plane, it’s important to note the restriction to $y > 0$ for $G$.

$$G(x, y; x_0, y_0) = \frac{1}{2\pi} \ln \sqrt{(x - x_0)^2 + (y - y_0)^2} - \frac{1}{2\pi} \ln \sqrt{(x - x_0)^2 + (y + y_0)^2}$$

$$= \frac{1}{2\pi} [\ln \sqrt{(x - x_0)^2 + (y - y_0)^2} - \ln \sqrt{(x - x_0)^2 + (y + y_0)^2}]$$

$$= \frac{1}{2\pi} \ln \frac{\sqrt{(x - x_0)^2 + (y - y_0)^2}}{\sqrt{(x - x_0)^2 + (y + y_0)^2}}$$

$$= \frac{1}{2\pi} \ln \sqrt{(x - x_0)^2 + (y - y_0)^2}$$

Therefore, the Green’s function for the upper half-plane is

$$G(x, y; x_0, y_0) = \frac{1}{4\pi} \ln \frac{(x - x_0)^2 + (y - y_0)^2}{(x - x_0)^2 + (y + y_0)^2}.$$
Now calculate the derivative of $G$ with respect to $y_0$.

$$\frac{\partial G}{\partial y_0} = \frac{1}{4\pi} \frac{(x-x_0)^2 + (y+y_0)^2}{(x-x_0)^2 + (y-y_0)^2} \frac{\partial}{\partial y_0} \left[ \frac{(x-x_0)^2 + (y-y_0)^2}{(x-x_0)^2 + (y+y_0)^2} \right]$$

$$= \frac{1}{4\pi} \frac{(x-x_0)^2 + (y+y_0)^2}{(x-x_0)^2 + (y-y_0)^2} \left\{ \frac{-2(y-y_0)[(x-x_0)^2 + (y+y_0)^2] - 2(y+y_0)[(x-x_0)^2 + (y-y_0)^2]}{[(x-x_0)^2 + (y+y_0)^2]^2} \right\}$$

$$= \frac{1}{4\pi} \frac{(x-x_0)^2 + (y+y_0)^2}{(x-x_0)^2 + (y-y_0)^2} \left\{ \frac{-4y[(x-x_0)^2 + (y-y_0)(y+y_0)]}{[(x-x_0)^2 + (y+y_0)^2]^2} \right\}$$

$$= -\frac{y}{\pi} \frac{(x-x_0)^2 + (y-y_0)(y+y_0)}{(x-x_0)^2 + (y+y_0)^2}$$

Evaluate it at $y_0 = 0$.

$$\left. \frac{\partial G}{\partial y_0} \right|_{y_0=0} = -\frac{y}{\pi} \frac{[(x-x_0)^2 + y^2]}{(x-x_0)^2 + y^2}$$

$$= -\frac{y}{\pi} \frac{1}{(x-x_0)^2 + y^2}$$

Therefore, the solution to Poisson’s equation is

$$u(x, y) = \int_0^\infty \int_{-\infty}^\infty G(x, y; x_0, y_0) f(x_0, y_0) \, dx_0 \, dy_0 - \int_{-\infty}^\infty h(x_0) \left. \frac{\partial G}{\partial y_0} \right|_{y_0=0} \, dx_0$$

$$= \frac{1}{4\pi} \int_0^\infty \int_{-\infty}^\infty f(x_0, y_0) \ln \left( \frac{(x-x_0)^2 + (y-y_0)^2}{(x-x_0)^2 + (y+y_0)^2} \right) \, dx_0 \, dy_0 - \int_{-\infty}^\infty h(x_0) \left[ \frac{-y}{\pi} \frac{1}{(x-x_0)^2 + y^2} \right] \, dx_0$$

$$= \frac{1}{4\pi} \int_0^\infty \int_{-\infty}^\infty f(x_0, y_0) \ln \left( \frac{(x-x_0)^2 + (y-y_0)^2}{(x-x_0)^2 + (y+y_0)^2} \right) \, dx_0 \, dy_0 + \frac{y}{\pi} \int_{-\infty}^\infty \frac{h(x_0)}{(x-x_0)^2 + y^2} \, dx_0$$

For the special case that $f = 0$ and $h = 1$, this solution reduces to

$$u(x, y) = \frac{y}{\pi} \int_{-\infty}^\infty \frac{1}{(x-x_0)^2 + y^2} \, dx_0.$$

Make the trigonometric substitution,

$$x_0 - x = y \tan \phi_0 \quad \Rightarrow \quad (x_0 - x)^2 + y^2 = y^2 \sec^2 \phi_0$$

$$dx_0 = y \sec^2 \phi_0 \, d\phi_0.$$

As a result,

$$u(x, y) = \frac{y}{\pi} \int_{\tan^{-1}(\infty)}^{\tan^{-1}(-\infty)} \frac{1}{y^2 \sec^2 \phi_0 \, y \sec^2 \phi_0 \, d\phi_0}$$

$$= \frac{1}{\pi} \int_{\pi/2}^{\pi/2} d\phi_0$$

$$= \frac{1}{\pi} (\pi)$$

$$= 1.$$