# Exercise 6

(a) Find the Green's function for the half-plane $\{(x, y) : \; y > 0\}$.

(b) Use it to solve the Dirichlet problem in the half-plane with boundary values $h(x)$.

(c) Calculate the solution with $u(x, 0) = 1$.

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**Solution**

The aim is to solve Poisson's equation in the upper half plane that is subject to a boundary condition along $y = 0$.

$$\begin{aligned} \Delta u &= f(x, y), \qquad -\infty < x < \infty, \; y > 0 \\ u(x, 0) &= h(x) \end{aligned}$$

A Green's function representation for the solution can be obtained from Green's second identity in two dimensions,

$$\iint_D (u\Delta v - v\Delta u)\, dA = \int_{\text{bdy } D} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) ds,$$

which holds for any two functions, $u$ and $v$, over any domain and its boundary. Let $v$ be the Green's function: $v = G$.

$$\iint_D (u\Delta G - G\Delta u)\, dA = \int_{\text{bdy } D} \left(u\frac{\partial G}{\partial n} - G\frac{\partial u}{\partial n}\right) ds \tag{1}$$

If we require $G = G(x, y; x_0, y_0)$ to satisfy

$$\begin{aligned} &\Delta G = \delta(x - x_0)\delta(y - y_0), \qquad -\infty < x < \infty, \; y > 0 \\ &G = 0 \text{ at } y = 0, \end{aligned}$$

where $(x_0, y_0)$ is a point in the half-plane, then equation (1) becomes

$$\iint_D [u(x, y)\delta(x - x_0)\delta(y - y_0) - G(x, y; x_0, y_0)f(x, y)]\, dA = \int_{\text{bdy } D} \left(u\frac{\partial G}{\partial n} - (0)\frac{\partial u}{\partial n}\right) ds.$$

Since the domain is the upper half-plane, the outward unit normal vector is $\hat{\mathbf{n}} = -\hat{\mathbf{y}}$, which means the normal derivative is $\partial/\partial n = -\partial/\partial y$.

$$\int_0^\infty \int_{-\infty}^\infty [u(x, y)\delta(x - x_0)\delta(y - y_0) - G(x, y; x_0, y_0)f(x, y)]\, dx\, dy = \int_{-\infty}^\infty u(x, 0)\left(-\frac{\partial G}{\partial y}\right)\bigg|_{y=0} dx$$

$$u(x_0, y_0) - \int_0^\infty \int_{-\infty}^\infty G(x, y; x_0, y_0)f(x, y)\, dx\, dy = \int_{-\infty}^\infty h(x)\left(-\frac{\partial G}{\partial y}\right)\bigg|_{y=0} dx$$

$$u(x_0, y_0) = \int_0^\infty \int_{-\infty}^\infty G(x, y; x_0, y_0)f(x, y)\, dx\, dy - \int_{-\infty}^\infty h(x)\frac{\partial G}{\partial y}\bigg|_{y=0} dx$$

Switch the roles of $x_0$ and $y_0$ with those of $x$ and $y$, respectively.

$$u(x, y) = \int_0^\infty \int_{-\infty}^\infty G(x_0, y_0; x, y)f(x_0, y_0)\, dx_0\, dy_0 - \int_{-\infty}^\infty h(x_0)\frac{\partial G}{\partial y_0}\bigg|_{y_0=0} dx_0$$

Therefore, using the fact that the Green's function is symmetric,

$$u(x,y)=\int_0^\infty\int_{-\infty}^\infty G(x,y;x_0,y_0)f(x_0,y_0)\,dx_0\,dy_0-\int_{-\infty}^\infty h(x_0)\frac{\partial G}{\partial y_0}\bigg|_{y_0=0}dx_0.$$

The solution for Poisson's equation is known, then, if the Green's function in the upper half-plane can be determined. Begin by finding the Green's function in the whole plane (no boundaries).

$$\Delta g=\delta(x-x_0)\delta(y-y_0),\qquad -\infty<x<\infty,\ -\infty<y<\infty$$

$g$ can be interpreted as the electrostatic potential, and $\delta(x-x_0)\delta(y-y_0)$ can be interpreted as the charge density for a unit charge located at $(x_0,y_0)$. Since there are no boundaries, $g$ is expected to vary solely as a function of the radial distance from $(x_0,y_0)$: $g=g(\mathscr{r})$, where $\mathscr{r}=\sqrt{(x-x_0)^2+(y-y_0)^2}$. Integrate both sides over a disk centered at $(x_0,y_0)$ with radius $\mathscr{r}$.

$$\iint\limits_{(x-x_0)^2+(y-y_0)^2\le\mathscr{r}^2}\Delta g\,dA=\iint\limits_{(x-x_0)^2+(y-y_0)^2\le\mathscr{r}^2}\delta(x-x_0)\delta(y-y_0)\,dA$$

Since the disk contains $(x_0,y_0)$, the right side is 1. Write the Laplacian operator as $\Delta=\nabla^2$

$$\iint\limits_{(x-x_0)^2+(y-y_0)^2\le\mathscr{r}^2}\nabla^2 g\,dA=1$$

$$\iint\limits_{(x-x_0)^2+(y-y_0)^2\le\mathscr{r}^2}\nabla\cdot\nabla g\,dA=1$$

and apply the two-dimensional divergence theorem.

$$\int\limits_{(x-x_0)^2+(y-y_0)^2=\mathscr{r}^2}\nabla g\cdot\hat{\mathscr{r}}\,ds=1$$

Here $\hat{\mathscr{r}}$ is the unit vector normal to this disk at every point on the boundary.

$$\int\limits_{(x-x_0)^2+(y-y_0)^2=\mathscr{r}^2}\frac{dg}{d\mathscr{r}}\,ds=1$$

Because $g$ only depends on $\mathscr{r}$, its derivative is constant on the disk's boundary.

$$\frac{dg}{d\mathscr{r}}\int\limits_{(x-x_0)^2+(y-y_0)^2=\mathscr{r}^2}ds=1$$

This line integral is just the disk's circumference.

$$\frac{dg}{d\mathscr{r}}(2\pi\mathscr{r})=1$$

Divide both sides by $2\pi\mathscr{r}$.

$$\frac{dg}{d\mathscr{r}}=\frac{1}{2\pi\mathscr{r}}$$

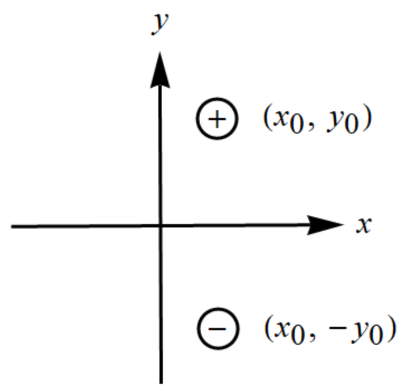
Integrate both sides with respect to $\mathscr{r}$.

$$g(\mathscr{r}) = \frac{1}{2\pi} \ln \mathscr{r}$$

The Green's function for the whole plane is then

$$g(x, y; x_0, y_0) = \frac{1}{2\pi} \ln \sqrt{(x - x_0)^2 + (y - y_0)^2}.$$

Now that it's known, the Green's function for the upper half-plane can be determined by the method of images. A convocation of point charges in the whole plane will be arranged so that the boundary condition, $G = 0$ along $y = 0$, is satisfied.




Since the two charges have the same magnitude but opposite polarity and are equally spaced from every point along the $x$-axis, the potential at every point on the $x$-axis is zero. The half-plane Green's function can now be written.

$$G(x, y; x_0, y_0) = +g(x, y; x_0, y_0) - g(x, y; x_0, -y_0), \quad y > 0$$

Since $g$ is defined over the whole plane, it's important to note the restriction to $y > 0$ for $G$.

$$\begin{aligned} G(x, y; x_0, y_0) &= \frac{1}{2\pi} \ln \sqrt{(x - x_0)^2 + (y - y_0)^2} - \frac{1}{2\pi} \ln \sqrt{(x - x_0)^2 + (y + y_0)^2} \\ &= \frac{1}{2\pi} [\ln \sqrt{(x - x_0)^2 + (y - y_0)^2} - \ln \sqrt{(x - x_0)^2 + (y + y_0)^2}] \\ &= \frac{1}{2\pi} \ln \frac{\sqrt{(x - x_0)^2 + (y - y_0)^2}}{\sqrt{(x - x_0)^2 + (y + y_0)^2}} \\ &= \frac{1}{2\pi} \ln \sqrt{\frac{(x - x_0)^2 + (y - y_0)^2}{(x - x_0)^2 + (y + y_0)^2}} \end{aligned}$$

Therefore, the Green's function for the upper half-plane is

$$G(x, y; x_0, y_0) = \frac{1}{4\pi} \ln \frac{(x - x_0)^2 + (y - y_0)^2}{(x - x_0)^2 + (y + y_0)^2}.$$

Now calculate the derivative of $G$ with respect to $y_0$.

$$\begin{aligned}
\frac{\partial G}{\partial y_0} &= \frac{1}{4\pi}\frac{(x-x_0)^2+(y+y_0)^2}{(x-x_0)^2+(y-y_0)^2}\frac{\partial}{\partial y_0}\left[\frac{(x-x_0)^2+(y-y_0)^2}{(x-x_0)^2+(y+y_0)^2}\right] \\
&= \frac{1}{4\pi}\frac{(x-x_0)^2+(y+y_0)^2}{(x-x_0)^2+(y-y_0)^2}\left\{\frac{[-2(y-y_0)][(x-x_0)^2+(y+y_0)^2]-[2(y+y_0)][(x-x_0)^2+(y-y_0)^2]}{[(x-x_0)^2+(y+y_0)^2]^2}\right\} \\
&= \frac{1}{4\pi}\frac{(x-x_0)^2+(y+y_0)^2}{(x-x_0)^2+(y-y_0)^2}\left\{\frac{-4y[(x-x_0)^2+(y-y_0)(y+y_0)]}{[(x-x_0)^2+(y+y_0)^2]^2}\right\} \\
&= -\frac{y}{\pi}\frac{(x-x_0)^2+(y-y_0)(y+y_0)}{[(x-x_0)^2+(y-y_0)^2][(x-x_0)^2+(y+y_0)^2]}
\end{aligned}$$

Evaluate it at $y_0 = 0$.

$$\begin{aligned}
\left.\frac{\partial G}{\partial y_0}\right|_{y_0=0} &= -\frac{y}{\pi}\frac{[(x-x_0)^2+y^2]}{[(x-x_0)^2+y^2][(x-x_0)^2+y^2]} \\
&= -\frac{y}{\pi}\frac{1}{(x-x_0)^2+y^2}
\end{aligned}$$

Therefore, the solution to Poisson's equation is

$$\begin{aligned}
u(x,y) &= \int_0^\infty\int_{-\infty}^\infty G(x,y;x_0,y_0)f(x_0,y_0)\,dx_0\,dy_0 - \int_{-\infty}^\infty h(x_0)\left.\frac{\partial G}{\partial y_0}\right|_{y_0=0} dx_0 \\
&= \int_0^\infty\int_{-\infty}^\infty\left[\frac{1}{4\pi}\ln\frac{(x-x_0)^2+(y-y_0)^2}{(x-x_0)^2+(y+y_0)^2}\right]f(x_0,y_0)\,dx_0\,dy_0 - \int_{-\infty}^\infty h(x_0)\left[-\frac{y}{\pi}\frac{1}{(x-x_0)^2+y^2}\right]dx_0 \\
&= \frac{1}{4\pi}\int_0^\infty\int_{-\infty}^\infty f(x_0,y_0)\ln\frac{(x-x_0)^2+(y-y_0)^2}{(x-x_0)^2+(y+y_0)^2}\,dx_0\,dy_0 + \frac{y}{\pi}\int_{-\infty}^\infty\frac{h(x_0)}{(x-x_0)^2+y^2}\,dx_0 \\
&= \frac{1}{4\pi}\int_0^\infty\int_{-\infty}^\infty f(x_0,y_0)\ln\frac{(x_0-x)^2+(y_0-y)^2}{(x_0-x)^2+(y_0+y)^2}\,dx_0\,dy_0 + \frac{y}{\pi}\int_{-\infty}^\infty\frac{h(x_0)}{(x_0-x)^2+y^2}\,dx_0.
\end{aligned}$$

For the special case that $f = 0$ and $h = 1$, this solution reduces to

$$u(x,y) = \frac{y}{\pi}\int_{-\infty}^\infty\frac{1}{(x_0-x)^2+y^2}\,dx_0.$$

Make the trigonometric substitution,

$$\begin{aligned}
x_0 - x &= y\tan\phi_0 \quad\Rightarrow\quad (x_0-x)^2+y^2 = y^2\sec^2\phi_0 \\
dx_0 &= y\sec^2\phi_0\,d\phi_0.
\end{aligned}$$

As a result,

$$\begin{aligned}
u(x,y) &= \frac{y}{\pi}\int_{\tan^{-1}(-\infty)}^{\tan^{-1}(\infty)}\frac{1}{y^2\sec^2\phi_0}\,y\sec^2\phi_0\,d\phi_0 \\
&= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}d\phi_0 \\
&= \frac{1}{\pi}(\pi) \\
&= 1.
\end{aligned}$$