Exercise 8

(a) Use Exercise 7 to find the harmonic function in the half-plane $\{y > 0\}$ with the boundary data $h(x) = 1$ for $x > 0$, $h(x) = 0$ for $x < 0$.

(b) Do the same as part (a) for the boundary data $h(x) = 1$ for $x > a$, $h(x) = 0$ for $x < a$. (*Hint:* Translate the preceding answer.)

(c) Use part (b) to solve the same problem with the boundary data $h(x)$, where $h(x)$ is any step function. That is,

$$h(x) = c_j \quad \text{for } a_{j-1} < x < a_j \quad \text{for } 1 \leq j \leq n,$$

where $-\infty = a_0 < a_1 < \cdots < a_{n-1} < a_n = \infty$ and the c_j are constants.

Solution

For the Laplace equation in the upper half-plane,

$$\begin{aligned} &\Delta u = 0, \qquad -\infty < x < \infty, \ y > 0 \\ &u(x, 0) = h(x), \end{aligned}$$

the solution developed in Exercise 6 reduces to

$$u(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{h(x_0)}{(x_0 - x)^2 + y^2} \, dx_0.$$

If

$$h(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases},$$

then

$$u(x, y) = \frac{y}{\pi} \int_{0}^{\infty} \frac{1}{(x_0 - x)^2 + y^2} \, dx_0.$$

Make the trigonometric substitution,

$$\begin{aligned} x_0 - x &= y \tan \phi_0 \quad \Rightarrow \quad (x_0 - x)^2 + y^2 = y^2 \sec^2 \phi_0 \\ dx_0 &= y \sec^2 \phi_0 \, d\phi_0. \end{aligned}$$

As a result,

$$\begin{aligned} u(x, y) &= \frac{y}{\pi} \int_{\tan^{-1}(-x/y)}^{\tan^{-1}(\infty)} \frac{1}{y^2 \sec^2 \phi_0} \, y \sec^2 \phi_0 \, d\phi_0 \\ &= \frac{1}{\pi} \int_{\tan^{-1}(-x/y)}^{\tan^{-1}(\infty)} d\phi_0 \\ &= \frac{1}{\pi} \left[\tan^{-1}(\infty) - \tan^{-1}\left(-\frac{x}{y}\right) \right] \\ &= \frac{1}{\pi} \left[\frac{\pi}{2} + \tan^{-1}\left(\frac{x}{y}\right) \right] \\ &= \frac{1}{2} + \frac{1}{\pi} \tan^{-1}\left(\frac{x}{y}\right). \end{aligned}$$

If

$$h(x) = \begin{cases} 1 & x > a \\ 0 & x < a \end{cases},$$

then

$$u(x,y) = \frac{y}{\pi}\int_a^\infty \frac{1}{(x_0-x)^2+y^2}\,dx_0.$$

Make the trigonometric substitution,

$$\begin{aligned} x_0 - x &= y\tan\phi_0 \quad \Rightarrow \quad (x_0-x)^2+y^2 = y^2\sec^2\phi_0 \\ dx_0 &= y\sec^2\phi_0\,d\phi_0. \end{aligned}$$

As a result,

$$\begin{aligned} u(x,y) &= \frac{y}{\pi}\int_{\tan^{-1}[(a-x)/y]}^{\tan^{-1}(\infty)} \frac{1}{y^2\sec^2\phi_0}\,y\sec^2\phi_0\,d\phi_0 \\ &= \frac{1}{\pi}\int_{\tan^{-1}[(a-x)/y]}^{\tan^{-1}(\infty)} d\phi_0 \\ &= \frac{1}{\pi}\left[\tan^{-1}(\infty) - \tan^{-1}\left(\frac{a-x}{y}\right)\right] \\ &= \frac{1}{\pi}\left[\frac{\pi}{2} + \tan^{-1}\left(\frac{x-a}{y}\right)\right] \\ &= \frac{1}{2} + \frac{1}{\pi}\tan^{-1}\left(\frac{x-a}{y}\right). \end{aligned}$$

If

$$h(x) = \begin{cases} c_1 & a_0 < x < a_1 \\ c_2 & a_1 < x < a_2 \\ \vdots & \vdots \\ c_{n-1} & a_{n-2} < x < a_{n-1} \\ c_n & a_{n-1} < x < a_n \end{cases},$$

then

$$\begin{aligned} u(x,y) &= \frac{y}{\pi}\left[\int_{a_0}^{a_1} \frac{c_1}{(x_0-x)^2+y^2}\,dx_0 + \int_{a_1}^{a_2} \frac{c_2}{(x_0-x)^2+y^2}\,dx_0 + \cdots \right. \\ &\qquad \left. + \int_{a_{n-2}}^{a_{n-1}} \frac{c_{n-1}}{(x_0-x)^2+y^2}\,dx_0 + \int_{a_{n-1}}^{a_n} \frac{c_n}{(x_0-x)^2+y^2}\,dx_0\right] \\ &= \frac{y}{\pi}\left[c_1\int_{-\infty}^{a_1} \frac{dx_0}{(x_0-x)^2+y^2} + c_2\int_{a_1}^{a_2} \frac{dx_0}{(x_0-x)^2+y^2} + \cdots \right. \\ &\qquad \left. + c_{n-1}\int_{a_{n-2}}^{a_{n-1}} \frac{dx_0}{(x_0-x)^2+y^2} + c_n\int_{a_{n-1}}^{\infty} \frac{dx_0}{(x_0-x)^2+y^2}\right]. \end{aligned}$$

Make the trigonometric substitution,

$$\begin{aligned} x_0 - x = y\tan\phi_0 \quad &\Rightarrow \quad (x_0 - x)^2 + y^2 = y^2\sec^2\phi_0 \\ dx_0 = y\sec^2\phi_0\,d\phi_0, \end{aligned}$$

in each of the integrals.

$$\begin{aligned} u(x,y) &= \frac{y}{\pi}\left[c_1\int_{\tan^{-1}(-\infty)}^{\tan^{-1}[(a_1-x)/y]}\frac{y\sec^2\phi_0\,d\phi_0}{y^2\sec^2\phi_0} + c_2\int_{\tan^{-1}[(a_1-x)/y]}^{\tan^{-1}[(a_2-x)/y]}\frac{y\sec^2\phi_0\,d\phi_0}{y^2\sec^2\phi_0} + \cdots\right. \\ &\qquad \left. + c_{n-1}\int_{\tan^{-1}[(a_{n-2}-x)/y]}^{\tan^{-1}[(a_{n-1}-x)/y]}\frac{y\sec^2\phi_0\,d\phi_0}{y^2\sec^2\phi_0} + c_n\int_{\tan^{-1}[(a_{n-1}-x)/y]}^{\tan^{-1}(\infty)}\frac{y\sec^2\phi_0\,d\phi_0}{y^2\sec^2\phi_0}\right] \\ &= \frac{1}{\pi}\left[c_1\int_{\tan^{-1}(-\infty)}^{\tan^{-1}[(a_1-x)/y]}d\phi_0 + c_2\int_{\tan^{-1}[(a_1-x)/y]}^{\tan^{-1}[(a_2-x)/y]}d\phi_0 + \cdots\right. \\ &\qquad \left. + c_{n-1}\int_{\tan^{-1}[(a_{n-2}-x)/y]}^{\tan^{-1}[(a_{n-1}-x)/y]}d\phi_0 + c_n\int_{\tan^{-1}[(a_{n-1}-x)/y]}^{\tan^{-1}(\infty)}d\phi_0\right] \\ &= \frac{1}{\pi}\left\{c_1\left[\tan^{-1}\left(\frac{a_1-x}{y}\right) - \tan^{-1}(-\infty)\right] + c_2\left[\tan^{-1}\left(\frac{a_2-x}{y}\right) - \tan^{-1}\left(\frac{a_1-x}{y}\right)\right] + \cdots\right. \\ &\qquad \left. + c_{n-1}\left[\tan^{-1}\left(\frac{a_{n-1}-x}{y}\right) - \tan^{-1}\left(\frac{a_{n-2}-x}{y}\right)\right] + c_n\left[\tan^{-1}(\infty) - \tan^{-1}\left(\frac{a_{n-1}-x}{y}\right)\right]\right\} \\ &= \frac{1}{\pi}\left\{c_1\left[\tan^{-1}\left(\frac{a_1-x}{y}\right) - \left(-\frac{\pi}{2}\right)\right] + c_2\left[\tan^{-1}\left(\frac{a_2-x}{y}\right) - \tan^{-1}\left(\frac{a_1-x}{y}\right)\right] + \cdots\right. \\ &\qquad \left. + c_{n-1}\left[\tan^{-1}\left(\frac{a_{n-1}-x}{y}\right) - \tan^{-1}\left(\frac{a_{n-2}-x}{y}\right)\right] + c_n\left[\left(\frac{\pi}{2}\right) - \tan^{-1}\left(\frac{a_{n-1}-x}{y}\right)\right]\right\} \\ &= \frac{1}{\pi}\left[\frac{\pi}{2}c_1 + (c_1 - c_2)\tan^{-1}\left(\frac{a_1-x}{y}\right) + (c_2 - c_3)\tan^{-1}\left(\frac{a_2-x}{y}\right) + \cdots\right. \\ &\qquad \left. + (c_{n-1} - c_n)\tan^{-1}\left(\frac{a_{n-1}-x}{y}\right) + \frac{\pi}{2}c_n\right] \end{aligned}$$

Therefore,

$$u(x,y) = \frac{1}{2}(c_1 + c_n) + \frac{1}{\pi}\sum_{i=2}^{n}(c_{i-1} - c_i)\tan^{-1}\left(\frac{a_{i-1}-x}{y}\right).$$

Part (a)

$$\Delta u = 0, \qquad -\infty < x < \infty,\ y > 0$$
$$u(x,0) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$

Because the boundary condition is defined piecewise for $x < 0$ and $x > 0$, we know from Exercise 7 that the solution to the half-plane problem will be a function of x/y: $u(x,y) = f(x/y)$. Substituting this into the Laplace equation results in an ODE for f,

$$(1+z^2)f''(z) = -2zf'(z),$$

which yields

$$f(z) = C_1 \tan^{-1} z + C_2$$

and

$$u(x,y) = C_1 \tan^{-1}\left(\frac{x}{y}\right) + C_2.$$

Take the limit of u as $y \to 0$.

$$\lim_{y\to 0} u(x,y) = C_1 \lim_{y\to 0} \tan^{-1}\left(\frac{x}{y}\right) + C_2$$

The sign of x is important in this limit because $\tan^{-1}(\pm\infty) = \pm\pi/2$.

$$\lim_{y\to 0} u(x,y) = \begin{cases} C_1\left(\frac{\pi}{2}\right) + C_2 & x > 0 \\ C_1\left(-\frac{\pi}{2}\right) + C_2 & x < 0 \end{cases}$$

Apply the boundary condition to determine C_1 and C_2.

$$C_1\left(\frac{\pi}{2}\right) + C_2 = 1$$
$$C_1\left(-\frac{\pi}{2}\right) + C_2 = 0$$

Solving this system of equations yields $C_1 = 1/\pi$ and $C_2 = 1/2$. Therefore,

$$u(x,y) = \frac{1}{\pi}\tan^{-1}\left(\frac{x}{y}\right) + \frac{1}{2}.$$

Part (b)

$$\Delta u = 0, \qquad -\infty < x < \infty,\ y > 0$$
$$u(x,0) = \begin{cases} 1 & x > a \\ 0 & x < a \end{cases}$$

Since the Laplace equation is translationally invariant, this problem can be solved using the result of part (a). Make the change of variables, $\bar{x} = x - a$.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad (-\infty < \bar{x} + a < \infty,\ y > 0)$$
$$u(\bar{x},0) = \begin{cases} 1 & \bar{x} + a > a \\ 0 & \bar{x} + a < a \end{cases}$$

Use the chain rule to write the derivative of u with respect to x in terms of this new variable.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \bar{x}}\frac{\partial \bar{x}}{\partial x} = \frac{\partial u}{\partial \bar{x}}(1) = \frac{\partial u}{\partial \bar{x}}$$
$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial \bar{x}}{\partial x}\frac{\partial}{\partial \bar{x}}\left(\frac{\partial u}{\partial \bar{x}}\right) = (1)\frac{\partial^2 u}{\partial \bar{x}^2} = \frac{\partial^2 u}{\partial \bar{x}^2}$$

The boundary value problem to solve is then

$$\frac{\partial^2 u}{\partial \bar{x}^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad -\infty < \bar{x} < \infty,\ y > 0$$
$$u(\bar{x},0) = \begin{cases} 1 & \bar{x} > 0 \\ 0 & \bar{x} < 0, \end{cases}$$

which is the same as before in part (a).

$$u(\bar{x},y) = \frac{1}{\pi}\tan^{-1}\left(\frac{\bar{x}}{y}\right) + \frac{1}{2}$$

Therefore, changing back to x,

$$u(x,y) = \frac{1}{\pi}\tan^{-1}\left(\frac{x-a}{y}\right) + \frac{1}{2}.$$

Part (c)

$$\Delta u = 0, \qquad -\infty < x < \infty, \ y > 0$$
$$u(x,0) = \begin{cases} c_1 & a_0 < x < a_1 \\ c_2 & a_1 < x < a_2 \\ \vdots & \vdots \\ c_{n-1} & a_{n-2} < x < a_{n-1} \\ c_n & a_{n-1} < x < a_n \end{cases}$$

In order to solve this problem, we will take advantage of the fact that the Laplace equation is linear. Let $u = u_1 + u_2 + \cdots + u_n$, where

$$\Delta u_1 = 0, \qquad -\infty < x < \infty, \ y > 0$$
$$u(x,0) = \begin{cases} c_1 & a_0 < x < a_1 \\ 0 & \text{otherwise} \end{cases} = c_1 H(-x + a_1)$$

$$\Delta u_2 = 0, \qquad -\infty < x < \infty, \ y > 0$$
$$u(x,0) = \begin{cases} c_2 & a_1 < x < a_2 \\ 0 & \text{otherwise} \end{cases} = c_2[H(x - a_1) - H(x - a_2)]$$

$$\vdots$$

$$\Delta u_{n-1} = 0, \qquad -\infty < x < \infty, \ y > 0$$
$$u(x,0) = \begin{cases} c_{n-1} & a_{n-2} < x < a_{n-1} \\ 0 & \text{otherwise} \end{cases} = c_{n-1}[H(x - a_{n-2}) - H(x - a_{n-1})]$$

$$\Delta u_n = 0, \qquad -\infty < x < \infty, \ y > 0$$
$$u(x,0) = \begin{cases} c_n & a_{n-1} < x < a_n \\ 0 & \text{otherwise} \end{cases} = c_n H(x - a_{n-1})$$

For convenience, each of the boundary conditions has been written in terms of the Heaviside function $H(x)$, which is defined as

$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases} \quad \rightarrow \quad H(x-a) = \begin{cases} 1 & x > a \\ 0 & x < a \end{cases} \quad \rightarrow \quad H(-x+a) = \begin{cases} 1 & x < a \\ 0 & x > a \end{cases}.$$

Noting that the problem in part (b) can be written as

$$\Delta v = 0, \qquad -\infty < x < \infty, \ y > 0$$
$$v(x,0) = H(x - a)$$

and that its solution is

$$v(x,y) = \frac{1}{\pi}\tan^{-1}\left(\frac{x-a}{y}\right) + \frac{1}{2},$$

the solutions for u_1, u_2, $\ldots$, u_n are linear combinations of v.

$$\begin{aligned}
u_1(x,y) &= c_1\left[\frac{1}{\pi}\tan^{-1}\left(\frac{-x+a_1}{y}\right)+\frac{1}{2}\right] \\
u_2(x,y) &= c_2\left\{\left[\frac{1}{\pi}\tan^{-1}\left(\frac{x-a_1}{y}\right)+\frac{1}{2}\right]-\left[\frac{1}{\pi}\tan^{-1}\left(\frac{x-a_2}{y}\right)+\frac{1}{2}\right]\right\} \\
&\vdots \\
u_{n-1}(x,y) &= c_{n-1}\left\{\left[\frac{1}{\pi}\tan^{-1}\left(\frac{x-a_{n-2}}{y}\right)+\frac{1}{2}\right]-\left[\frac{1}{\pi}\tan^{-1}\left(\frac{x-a_{n-1}}{y}\right)+\frac{1}{2}\right]\right\} \\
u_n(x,y) &= c_n\left[\frac{1}{\pi}\tan^{-1}\left(\frac{x-a_{n-1}}{y}\right)+\frac{1}{2}\right]
\end{aligned}$$

Rewrite these formulas.

$$\begin{aligned}
u_1(x,y) &= \frac{c_1}{\pi}\tan^{-1}\left(\frac{a_1-x}{y}\right)+\frac{c_1}{2} \\
u_2(x,y) &= -\frac{c_2}{\pi}\tan^{-1}\left(\frac{a_1-x}{y}\right)+\frac{c_2}{\pi}\tan^{-1}\left(\frac{a_2-x}{y}\right) \\
&\vdots \\
u_{n-1}(x,y) &= -\frac{c_{n-1}}{\pi}\tan^{-1}\left(\frac{a_{n-2}-x}{y}\right)+\frac{c_{n-1}}{\pi}\tan^{-1}\left(\frac{a_{n-1}-x}{y}\right) \\
u_n(x,y) &= -\frac{c_n}{\pi}\tan^{-1}\left(\frac{a_{n-1}-x}{y}\right)+\frac{c_n}{2}
\end{aligned}$$

Therefore,

$$\begin{aligned}
u(x,y) &= u_1+u_2+\cdots+u_n \\
&= \frac{1}{2}(c_1+c_n)+\frac{1}{\pi}\left[(c_1-c_2)\tan^{-1}\left(\frac{a_1-x}{y}\right)+(c_2-c_3)\tan^{-1}\left(\frac{a_2-x}{y}\right)+\cdots\right. \\
&\qquad\left.+\,(c_{n-1}-c_n)\tan^{-1}\left(\frac{a_{n-1}-x}{y}\right)\right] \\
&= \frac{1}{2}(c_1+c_n)+\frac{1}{\pi}\sum_{i=2}^{n}(c_{i-1}-c_i)\tan^{-1}\left(\frac{a_{i-1}-x}{y}\right).
\end{aligned}$$