Exercise 10

Verify the formula (11) for $G(\mathbf{x},\mathbf{0})$, the Green's function with its second argument at the center of the sphere.

Solution

The aim here is to solve the Poisson equation inside a solid ball with radius R that is subject to a boundary condition. Use a spherical coordinate system (ρ,ϕ,θ) in which θ is the angle from the polar axis.

$$\begin{aligned} &\Delta u = f(\rho,\phi,\theta), \quad \rho < R,\ 0<\phi<2\pi,\ 0<\theta<\pi \\ &u(R,\phi,\theta) = F(\phi,\theta) \end{aligned}$$

A Green's function representation for the solution can be obtained from Green's second identity,

$$\iiint_D (u\Delta v - v\Delta u)\,dV = \iint_{\text{bdy } D} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) dS,$$

which holds for any two functions, u and v, over any domain and its boundary. Let v be the Green's function: $v = G = G(\rho,\phi,\theta;\rho_0,\phi_0,\theta_0)$.

$$\iiint_D (u\Delta G - G\Delta u)\,dV = \iint_{\text{bdy } D} \left(u\frac{\partial G}{\partial n} - G\frac{\partial u}{\partial n}\right) dS \tag{1}$$

If we require it to satisfy

$$\begin{aligned} &\Delta G = \delta(\rho-\rho_0)\delta(\phi-\phi_0)\delta(\theta-\theta_0), \qquad \rho<R,\ 0<\phi<2\pi,\ 0<\theta<\pi \\ &G = 0 \text{ at } \rho = R, \end{aligned}$$

where (ρ_0,ϕ_0,θ_0) is a point in the ball, then equation (1) becomes

$$\iiint_D [u(\rho,\phi,\theta)\delta(\rho-\rho_0)\delta(\phi-\phi_0)\delta(\theta-\theta_0) - G(\rho,\phi,\theta;\rho_0,\phi_0,\theta_0)f(\rho,\phi,\theta)]\,dV$$
$$= \iint_{\text{bdy } D} \left(u\frac{\partial G}{\partial n} - (0)\frac{\partial u}{\partial n}\right) dS.$$

Since the domain is a ball centered at the origin, the outward unit normal vector is $\hat{\mathbf{n}} = \hat{\boldsymbol{\rho}}$, which means the normal derivative is $\partial/\partial n = \partial/\partial\rho$.

$$\iiint_D u(\rho,\phi,\theta)\delta(\rho-\rho_0)\delta(\phi-\phi_0)\delta(\theta-\theta_0)\,dV - \iiint_D G(\rho,\phi,\theta;\rho_0,\phi_0,\theta_0)f(\rho,\phi,\theta)\,dV = \iint_{\text{bdy } D} u\frac{\partial G}{\partial\rho}\,dS$$

The integral involving the delta functions is $u(\rho_0,\phi_0,\theta_0)$.

$$u(\rho_0,\phi_0,\theta_0) - \int_0^\pi\int_0^{2\pi}\int_0^R G(\rho,\phi,\theta;\rho_0,\phi_0,\theta_0)f(\rho,\phi,\theta)\rho^2\sin\theta\,d\rho\,d\phi\,d\theta$$
$$= \int_0^\pi\int_0^{2\pi} u(R,\phi,\theta)\left.\frac{\partial G}{\partial\rho}\right|_{\rho=R} R^2\sin\theta\,d\phi\,d\theta$$

Solve for u.

$$u(\rho_0,\phi_0,\theta_0) = \int_0^{\pi}\int_0^{2\pi}\int_0^{R} G(\rho,\phi,\theta;\rho_0,\phi_0,\theta_0) f(\rho,\phi,\theta)\rho^2 \sin\theta \, d\rho \, d\phi \, d\theta \\ + R^2 \int_0^{\pi}\int_0^{2\pi} F(\phi,\theta) \left.\frac{\partial G}{\partial \rho}\right|_{\rho=R} \sin\theta \, d\phi \, d\theta$$

Switch the roles of ρ_0, ϕ_0, and θ_0 with those of ρ, ϕ, and θ, respectively.

$$u(\rho,\phi,\theta) = \int_0^{\pi}\int_0^{2\pi}\int_0^{R} G(\rho_0,\phi_0,\theta_0;\rho,\phi,\theta) f(\rho_0,\phi_0,\theta_0)\rho_0^2 \sin\theta_0 \, d\rho_0 \, d\phi_0 \, d\theta_0 \\ + R^2 \int_0^{\pi}\int_0^{2\pi} F(\phi_0,\theta_0) \left.\frac{\partial G}{\partial \rho_0}\right|_{\rho_0=R} \sin\theta_0 \, d\phi_0 \, d\theta_0$$

Therefore, using the fact that the Green's function is symmetric,

$$u(\rho,\phi,\theta) = \int_0^{\pi}\int_0^{2\pi}\int_0^{R} G(\rho,\phi,\theta;\rho_0,\phi_0,\theta_0) f(\rho_0,\phi_0,\theta_0)\rho_0^2 \sin\theta_0 \, d\rho_0 \, d\phi_0 \, d\theta_0 \\ + R^2 \int_0^{\pi}\int_0^{2\pi} F(\phi_0,\theta_0) \left.\frac{\partial G}{\partial \rho_0}\right|_{\rho_0=R} \sin\theta_0 \, d\phi_0 \, d\theta_0.$$

The solution for Poisson's equation is known, then, if the Green's function inside the ball can be determined. Begin by finding the Green's function in infinite space (no boundaries).

$$\Delta g = \delta(x-x_0)\delta(y-y_0)\delta(z-z_0), \qquad -\infty < x,y,z < \infty$$

g can be interpreted as the electrostatic potential, and $\delta(x-x_0)\delta(y-y_0)\delta(z-z_0)$ can be interpreted as the charge density for a unit charge located at (x_0,y_0,z_0). Since there are no boundaries, g is expected to vary solely as a function of the radial distance from (x_0,y_0,z_0): $g = g(\mathscr{r})$, where $\mathscr{r} = \sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}$. Integrate both sides over a solid ball centered at (x_0,y_0,z_0) with radius $\mathscr{r}$.

$$\iiint\limits_{\substack{(x-x_0)^2+(y-y_0)^2\\+(z-z_0)^2\le \mathscr{r}^2}} \Delta g \, dV = \iiint\limits_{\substack{(x-x_0)^2+(y-y_0)^2\\+(z-z_0)^2\le \mathscr{r}^2}} \delta(x-x_0)\delta(y-y_0)\delta(z-z_0)\, dV$$

Since the ball contains (x_0,y_0,z_0), the right side is 1. Write the Laplacian operator Δ as ∇^2

$$\iiint\limits_{\substack{(x-x_0)^2+(y-y_0)^2\\+(z-z_0)^2\le \mathscr{r}^2}} \nabla^2 g \, dV = 1$$

$$\iiint\limits_{\substack{(x-x_0)^2+(y-y_0)^2\\+(z-z_0)^2\le \mathscr{r}^2}} \nabla\cdot\nabla g \, dV = 1$$

and apply the divergence theorem.

$$\iint\limits_{\substack{(x-x_0)^2+(y-y_0)^2\\+(z-z_0)^2= \mathscr{r}^2}} \nabla g\cdot \hat{\mathscr{r}} \, dS = 1$$

Here $\hat{\boldsymbol{\mathscr{r}}}$ is the unit vector normal to this ball at every point on the boundary.

$$\iint\limits_{\substack{(x-x_0)^2+(y-y_0)^2 \\ +(z-z_0)^2=\mathscr{r}^2}} \frac{dg}{d\mathscr{r}}\, dS = 1$$

Because g only depends on $\mathscr{r}$, its derivative is constant on the ball's boundary.

$$\frac{dg}{d\mathscr{r}} \iint\limits_{\substack{(x-x_0)^2+(y-y_0)^2 \\ +(z-z_0)^2=\mathscr{r}^2}} dS = 1$$

This surface integral is just the ball's surface area.

$$\frac{dg}{d\mathscr{r}}(4\pi\mathscr{r}^2) = 1$$

Divide both sides by $4\pi\mathscr{r}^2$.

$$\frac{dg}{d\mathscr{r}} = \frac{1}{4\pi\mathscr{r}^2}$$

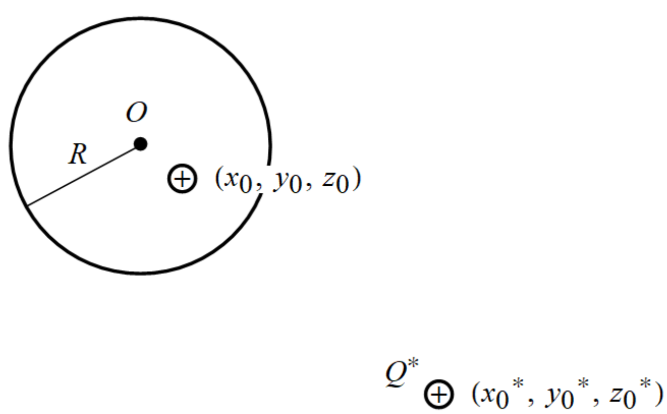
Integrate both sides with respect to $\mathscr{r}$.

$$g(\mathscr{r}) = -\frac{1}{4\pi\mathscr{r}}$$

The infinite-space Green's function is then

$$g(x,y,z;x_0,y_0,z_0) = -\frac{1}{4\pi\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}.$$

Now that it's known, the Green's function for the ball can be determined by the method of images. A convocation of point charges in infinite space will be arranged so that the boundary condition, $G = 0$ along $\rho = R$, is satisfied.



For a positive unit charge located at (x_0, y_0, z_0) inside the ball, place a charge Q^* at (x_0^*, y_0^*, z_0^*) outside the ball such that the charges are collinear with the origin. x_0^*, y_0^*, z_0^*, and Q^* are all unknown at the moment.

Write the Green's function in the ball (valid for $x^2 + y^2 + z^2 < R^2$).

$$\begin{aligned}
G(x, y, z; x_0, y_0, z_0) &= +g(x, y, z; x_0, y_0, z_0) + Q^* g(x, y, z; x_0^*, y_0^*, z_0^*) \\
&= -\frac{1}{4\pi\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}} - \frac{Q^*}{4\pi\sqrt{(x-x_0^*)^2 + (y-y_0^*)^2 + (z-z_0^*)^2}} \\
&= -\frac{1}{4\pi}\left[\frac{1}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}} + \frac{Q^*}{\sqrt{(x-x_0^*)^2 + (y-y_0^*)^2 + (z-z_0^*)^2}}\right] \\
&= -\frac{1}{4\pi}\left[\frac{1}{\sqrt{(x^2+y^2+z^2) + (x_0^2 + y_0^2 + z_0^2) - 2(xx_0 + yy_0 + zz_0)}}\right. \\
&\qquad \left. + \frac{Q^*}{\sqrt{(x^2+y^2+z^2) + (x_0^{*2} + y_0^{*2} + z_0^{*2}) - 2(xx_0^* + yy_0^* + zz_0^*)}}\right] \quad (2)
\end{aligned}$$

$$\begin{aligned}
G(\rho, \phi, \theta; \rho_0, \phi_0, \theta_0) &= -\frac{1}{4\pi}\left[\frac{1}{\sqrt{\rho^2 + \rho_0^2 - 2(\mathbf{x}\cdot\mathbf{x}_0)}} + \frac{Q^*}{\sqrt{\rho^2 + \rho_0^{*2} - 2(\mathbf{x}\cdot\mathbf{x}_0^*)}}\right] \\
&= -\frac{1}{4\pi}\left(\frac{1}{\sqrt{\rho^2 + \rho_0^2 - 2|\mathbf{x}||\mathbf{x}_0|\cos\alpha}} + \frac{Q^*}{\sqrt{\rho^2 + \rho_0^{*2} - 2|\mathbf{x}||\mathbf{x}_0^*|\cos\alpha}}\right) \\
&= -\frac{1}{4\pi}\left(\frac{1}{\sqrt{\rho^2 + \rho_0^2 - 2\rho\rho_0\cos\alpha}} + \frac{Q^*}{\sqrt{\rho^2 + \rho_0^{*2} - 2\rho\rho_0^*\cos\alpha}}\right) \\
&= -\frac{1}{4\pi}\left(\frac{1}{\rho\sqrt{1 + \frac{\rho_0^2}{\rho^2} - \frac{2\rho_0}{\rho}\cos\alpha}} + \frac{Q^*}{\rho_0^*\sqrt{\frac{\rho^2}{\rho_0^{*2}} + 1 - \frac{2\rho}{\rho_0^*}\cos\alpha}}\right)
\end{aligned}$$

Here α represents the angle between $\mathbf{x}$, the position vector of the point we're interested in knowing the potential at, and $\mathbf{x}_0$, the position vector of the positive unit charge. Because this positive unit charge and the image charge Q^* are collinear, α is also the angle between $\mathbf{x}$ and $\mathbf{x}_0^*$, the position vector of the image. The potential at $\rho = R$ is zero.

$$G(R, \phi, \theta; \rho_0, \phi_0, \theta_0) = -\frac{1}{4\pi}\left(\frac{1}{R\sqrt{1 + \frac{\rho_0^2}{R^2} - \frac{2\rho_0}{R}\cos\alpha}} + \frac{Q^*}{\rho_0^*\sqrt{1 + \frac{R^2}{\rho_0^{*2}} - \frac{2R}{\rho_0^*}\cos\alpha}}\right) = 0$$

In order for the quantity in parentheses to vanish, set

$$\frac{\rho_0}{R} = \frac{R}{\rho_0^*} \quad \text{and} \quad \frac{1}{R} + \frac{Q^*}{\rho_0^*} = 0,$$

which means

$$\begin{aligned}
\rho_0^* = \frac{R^2}{\rho_0} \quad \text{and} \quad Q^* &= -\frac{\rho_0^*}{R} \\
Q^* &= -\frac{R}{\rho_0}.
\end{aligned}$$

Now write equation (2) in terms of spherical coordinates and then substitute these formulas for ρ_0^* and Q^*.

$$\begin{aligned}
G(x,y,z;x_0,y_0,z_0) &= -\frac{1}{4\pi}\left[\frac{1}{\sqrt{(x^2+y^2+z^2)+(x_0^2+y_0^2+z_0^2)-2(xx_0+yy_0+zz_0)}}\right.\\
&\qquad \left.+\frac{Q^*}{\sqrt{(x^2+y^2+z^2)+(x_0^{*2}+y_0^{*2}+z_0^{*2})-2(xx_0^*+yy_0^*+zz_0^*)}}\right]\\
G(\rho,\phi,\theta;\rho_0,\phi_0,\theta_0) &= -\frac{1}{4\pi}\left\{\frac{1}{\sqrt{\rho^2+\rho_0^2-2[(\rho\cos\phi\sin\theta)(\rho_0\cos\phi_0\sin\theta_0)+(\rho\sin\phi\sin\theta)(\rho_0\sin\phi_0\sin\theta_0)+(\rho\cos\theta)(\rho_0\cos\theta_0)]}}\right.\\
&\qquad \left.+\frac{Q^*}{\sqrt{\rho^2+\rho_0^{*2}-2[(\rho\cos\phi\sin\theta)(\rho_0^*\cos\phi_0\sin\theta_0)+(\rho\sin\phi\sin\theta)(\rho_0^*\sin\phi_0\sin\theta_0)+(\rho\cos\theta)(\rho_0^*\cos\theta_0)]}}\right\}\\
&= -\frac{1}{4\pi}\left[\frac{1}{\sqrt{\rho^2+\rho_0^2-2\rho\rho_0(\cos\phi\cos\phi_0\sin\theta\sin\theta_0+\sin\phi\sin\phi_0\sin\theta\sin\theta_0+\cos\theta\cos\theta_0)}}\right.\\
&\qquad \left.+\frac{Q^*}{\sqrt{\rho^2+\rho_0^{*2}-2\rho\rho_0^*(\cos\phi\cos\phi_0\sin\theta\sin\theta_0+\sin\phi\sin\phi_0\sin\theta\sin\theta_0+\cos\theta\cos\theta_0)}}\right]\\
&= -\frac{1}{4\pi}\left[\frac{1}{\sqrt{\rho^2+\rho_0^2-2\rho\rho_0[\cos(\phi-\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]}}\right.\\
&\qquad \left.+\frac{Q^*}{\sqrt{\rho^2+\rho_0^{*2}-2\rho\rho_0^*[\cos(\phi-\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]}}\right]\\
&= -\frac{1}{4\pi}\left[\frac{1}{\sqrt{\rho^2+\rho_0^2-2\rho\rho_0[\cos(\phi-\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]}}\right.\\
&\qquad \left.+\frac{-\frac{R}{\rho_0}}{\sqrt{\rho^2+\frac{R^4}{\rho_0^2}-2\rho\frac{R^2}{\rho_0}[\cos(\phi-\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]}}\right]\\
&= -\frac{1}{4\pi}\left[\frac{1}{\sqrt{\rho^2+\rho_0^2-2\rho\rho_0[\cos(\phi-\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]}}\right.\\
&\qquad \left.-\frac{R}{\sqrt{\rho^2\rho_0^2+R^4-2R^2\rho\rho_0[\cos(\phi-\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]}}\right]
\end{aligned}$$

Therefore, the Green's function inside a ball of radius R in spherical coordinates is

$$G(\rho,\phi,\theta;\rho_0,\phi_0,\theta_0) = \frac{1}{4\pi}\left[\frac{R}{\sqrt{\rho^2\rho_0^2 + R^4 - 2R^2\rho\rho_0[\cos(\phi-\phi_0)\sin\theta\sin\theta_0 + \cos\theta\cos\theta_0]}} - \frac{1}{\sqrt{\rho^2 + \rho_0^2 - 2\rho\rho_0[\cos(\phi-\phi_0)\sin\theta\sin\theta_0 + \cos\theta\cos\theta_0]}}\right].$$

Observe that if the positive unit charge is placed at the center of the ball, then the potential at any point inside has a much simpler form.

$$\begin{aligned}\lim_{\rho_0\to 0} G(\rho,\phi,\theta;\rho_0,\phi_0,\theta_0) &= \frac{1}{4\pi}\left(\frac{R}{\sqrt{R^4}} - \frac{1}{\sqrt{\rho^2}}\right)\\ &= \frac{1}{4\pi}\left(\frac{1}{R} - \frac{1}{\rho}\right)\end{aligned}$$

This is essentially equation (11) in the textbook. Calculate $\partial G/\partial\rho_0$

$$\frac{\partial G}{\partial \rho_0} = \frac{1}{4\pi}\left\{-\frac{R}{2}\frac{2\rho^2\rho_0 - 2R^2\rho[\cos(\phi-\phi_0)\sin\theta\sin\theta_0 + \cos\theta\cos\theta_0]}{\left\{\rho^2\rho_0^2 + R^4 - 2R^2\rho\rho_0[\cos(\phi-\phi_0)\sin\theta\sin\theta_0 + \cos\theta\cos\theta_0]\right\}^{3/2}} + \frac{1}{2}\frac{2\rho_0 - 2\rho[\cos(\phi-\phi_0)\sin\theta\sin\theta_0 + \cos\theta\cos\theta_0]}{\left\{\rho^2 + \rho_0^2 - 2\rho\rho_0[\cos(\phi-\phi_0)\sin\theta\sin\theta_0 + \cos\theta\cos\theta_0]\right\}^{3/2}}\right\}$$

and then evaluate it at $\rho_0 = R$.

$$\begin{aligned}\left.\frac{\partial G}{\partial \rho_0}\right|_{\rho_0=R} &= \frac{1}{4\pi}\left\{-\frac{R}{2}\frac{2R\rho^2 - 2R^2\rho[\cos(\phi-\phi_0)\sin\theta\sin\theta_0 + \cos\theta\cos\theta_0]}{R^3\left\{\rho^2 + R^2 - 2R\rho[\cos(\phi-\phi_0)\sin\theta\sin\theta_0 + \cos\theta\cos\theta_0]\right\}^{3/2}} + \frac{1}{2}\frac{2R - 2\rho[\cos(\phi-\phi_0)\sin\theta\sin\theta_0 + \cos\theta\cos\theta_0]}{\left\{\rho^2 + R^2 - 2R\rho[\cos(\phi-\phi_0)\sin\theta\sin\theta_0 + \cos\theta\cos\theta_0]\right\}^{3/2}}\right\}\\ &= \frac{1}{4\pi}\frac{-\frac{\rho^2}{R} + R}{\left\{\rho^2 + R^2 - 2R\rho[\cos(\phi-\phi_0)\sin\theta\sin\theta_0 + \cos\theta\cos\theta_0]\right\}^{3/2}}\\ &= \frac{1}{4\pi R}\frac{R^2 - \rho^2}{\left\{\rho^2 + R^2 - 2R\rho[\cos(\phi-\phi_0)\sin\theta\sin\theta_0 + \cos\theta\cos\theta_0]\right\}^{3/2}}\end{aligned}$$

Therefore, the solution to the Poisson equation inside a ball with a prescribed boundary condition at $\rho = R$ is

$$\begin{aligned} u(\rho,\phi,\theta) = \frac{1}{4\pi}\int_0^\pi\int_0^{2\pi}\int_0^R \Bigg[&\frac{R}{\sqrt{\rho^2\rho_0^2 + R^4 - 2R^2\rho\rho_0[\cos(\phi-\phi_0)\sin\theta\sin\theta_0 + \cos\theta\cos\theta_0]}} \\ &- \frac{1}{\sqrt{\rho^2 + \rho_0^2 - 2\rho\rho_0[\cos(\phi-\phi_0)\sin\theta\sin\theta_0 + \cos\theta\cos\theta_0]}}\Bigg] f(\rho_0,\phi_0,\theta_0)\rho_0^2\sin\theta_0\,d\rho_0\,d\phi_0\,d\theta_0 \\ &+ \frac{R}{4\pi}\int_0^\pi\int_0^{2\pi}\frac{R^2-\rho^2}{\{\rho^2+R^2-2R\rho[\cos(\phi-\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]\}^{3/2}}F(\phi_0,\theta_0)\sin\theta_0\,d\phi_0\,d\theta_0. \end{aligned}$$

If $f = 0$, then the solution reduces to

$$u(\rho,\phi,\theta) = \frac{R}{4\pi}\int_0^\pi\int_0^{2\pi}\frac{R^2-\rho^2}{\{\rho^2+R^2-2R\rho[\cos(\phi-\phi_0)\sin\theta\sin\theta_0+\cos\theta\cos\theta_0]\}^{3/2}}F(\phi_0,\theta_0)\sin\theta_0\,d\phi_0\,d\theta_0,$$

which is essentially equation (16) in the textbook.