Exercise 10

Verify the formula (11) for \( G(\mathbf{x}, 0) \), the Green’s function with its second argument at the center of the sphere.

Solution

The aim here is to solve the Poisson equation inside a solid ball with radius \( R \) that is subject to a boundary condition. Use a spherical coordinate system \((\rho, \phi, \theta)\) in which \( \theta \) is the angle from the polar axis.

\[
\Delta u = f(\rho, \phi, \theta), \quad \rho < R, \quad 0 < \phi < 2\pi, \quad 0 < \theta < \pi
\]

\[u(R, \phi, \theta) = F(\phi, \theta)\]

A Green’s function representation for the solution can be obtained from Green’s second identity,

\[
\iiint_D (u \Delta v - v \Delta u) dV = \iint_{\partial D} \left( \frac{\partial v}{\partial n} - \frac{\partial u}{\partial n} \right) dS,
\]

which holds for any two functions, \( u \) and \( v \), over any domain and its boundary. Let \( v \) be the Green’s function:

\[v = G = G(\rho, \phi, \theta; \rho_0, \phi_0, \theta_0)\].

If we require it to satisfy

\[
\Delta G = \delta(\rho - \rho_0) \delta(\phi - \phi_0) \delta(\theta - \theta_0), \quad \rho < R, \quad 0 < \phi < 2\pi, \quad 0 < \theta < \pi
\]

\[G = 0 \text{ at } \rho = R\],

where \((\rho_0, \phi_0, \theta_0)\) is a point in the ball, then equation (1) becomes

\[
\iiint_D [u(\rho, \phi, \theta) \delta(\rho - \rho_0) \delta(\phi - \phi_0) \delta(\theta - \theta_0) - G(\rho, \phi, \theta; \rho_0, \phi_0, \theta_0) f(\rho, \phi, \theta)] dV
\]

\[= \iint_{\partial D} \left( \frac{\partial G}{\partial n} - (0) \frac{\partial u}{\partial n} \right) dS.\]

Since the domain is a ball centered at the origin, the outward unit normal vector is \( \hat{n} = \hat{\rho} \), which means the normal derivative is \( \partial/\partial n = \partial/\partial \rho \).

\[
\iiint_D u(\rho, \phi, \theta) \delta(\rho - \rho_0) \delta(\phi - \phi_0) \delta(\theta - \theta_0) dV - \iiint_D G(\rho, \phi, \theta; \rho_0, \phi_0, \theta_0) f(\rho, \phi, \theta) dV = \iint_{\partial D} \frac{\partial G}{\partial \rho} dS
\]

The integral involving the delta functions is \( u(\rho_0, \phi_0, \theta_0) \).

\[
u(\rho_0, \phi_0, \theta_0) = \int_0^{2\pi} \int_0^\pi R f(\rho, \phi, \theta) \rho^2 \sin \theta d\rho d\phi d\theta
\]

\[= \int_0^{2\pi} \int_0^\pi u(R, \phi, \theta) \frac{\partial G}{\partial \rho} \bigg|_{\rho=R} R^2 \sin \theta d\phi d\theta\]
Solve for \( u \).
\[
u(\rho_0, \phi_0, \theta_0) = \int_0^\pi \int_0^{2\pi} \int_0^R G(\rho, \phi, \theta; \rho_0, \phi_0, \theta_0) f(\rho, \phi, \theta) \rho^2 \sin \theta \, d\rho \, d\phi \, d\theta
\]
\[+ R^2 \int_0^\pi \int_0^{2\pi} F(\phi, \theta) \frac{\partial G}{\partial \rho} \bigg|_{\rho=R} \sin \theta \, d\phi \, d\theta \]

Switch the roles of \( \rho_0, \phi_0, \) and \( \theta_0 \) with those of \( \rho, \phi, \) and \( \theta, \) respectively.
\[
u(\rho, \phi, \theta) = \int_0^\pi \int_0^{2\pi} \int_0^R G(\rho, \phi_0, \theta_0; \rho, \phi, \theta) f(\rho, \phi_0, \theta_0) \rho_0^2 \sin \theta_0 \, d\rho_0 \, d\phi_0 \, d\theta_0
\]
\[+ R^2 \int_0^\pi \int_0^{2\pi} F(\phi_0, \theta_0) \frac{\partial G}{\partial \rho} \bigg|_{\rho_0=R} \sin \theta_0 \, d\phi_0 \, d\theta_0 \]

Therefore, using the fact that the Green’s function is symmetric,
\[
u(\rho, \phi, \theta) = \int_0^\pi \int_0^{2\pi} \int_0^R G(\rho, \phi, \theta; \rho_0, \phi_0, \theta_0) f(\rho_0, \phi_0, \theta_0) \rho^2 \sin \theta_0 \, d\rho_0 \, d\phi_0 \, d\theta_0
\]
\[+ R^2 \int_0^\pi \int_0^{2\pi} F(\phi_0, \theta_0) \frac{\partial G}{\partial \rho} \bigg|_{\rho_0=R} \sin \theta_0 \, d\phi_0 \, d\theta_0. \]

The solution for Poisson’s equation is known, then, if the Green’s function inside the ball can be determined. Begin by finding the Green’s function in infinite space (no boundaries).
\[\Delta g = \delta(x-x_0)\delta(y-y_0)\delta(z-z_0), \quad -\infty < x, y, z < \infty\]
g can be interpreted as the electrostatic potential, and \( \delta(x-x_0)\delta(y-y_0)\delta(z-z_0) \) can be interpreted as the charge density for a unit charge located at \( (x_0, y_0, z_0) \). Since there are no boundaries, \( g \) is expected to vary solely as a function of the radial distance from \( (x_0, y_0, z_0) \): \( g = g(\hat{r}) \), where \( \hat{r} = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2} \). Integrate both sides over a solid ball centered at \( (x_0, y_0, z_0) \) with radius \( \hat{r} \).
\[
\iiint_{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 \leq \hat{r}^2} \Delta g \, dV = \iiint_{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 \leq \hat{r}^2} \delta(x-x_0)\delta(y-y_0)\delta(z-z_0) \, dV
\]

Since the ball contains \( (x_0, y_0, z_0) \), the right side is 1. Write the Laplacian operator \( \Delta \) as \( \nabla^2 \)
\[
\iiint_{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 \leq \hat{r}^2} \nabla^2 g \, dV = 1
\]
\[
\iiint_{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 \leq \hat{r}^2} \nabla \cdot \nabla g \, dV = 1
\]

and apply the divergence theorem.
\[
\int_{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = \hat{r}^2} \nabla g \cdot \hat{n} \, dS = 1
\]

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Here \( \hat{\mathbf{r}} \) is the unit vector normal to this ball at every point on the boundary.

\[
\int \frac{dg}{d\mathbf{r}} dS = 1
\]

\[
\frac{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}{r^2} = 1
\]

Because \( g \) only depends on \( r \), its derivative is constant on the ball’s boundary.

\[
\frac{dg}{d\mathbf{r}} \int \frac{dS}{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2} = 1
\]

This surface integral is just the ball’s surface area.

\[
\frac{dg}{d\mathbf{r}} (4\pi r^2) = 1
\]

Divide both sides by \( 4\pi r^2 \).

\[
\frac{dg}{d\mathbf{r}} = \frac{1}{4\pi r^2}
\]

Integrate both sides with respect to \( r \).

\[
g(r) = -\frac{1}{4\pi r}
\]

The infinite-space Green’s function is then

\[
g(x, y, z; x_0, y_0, z_0) = -\frac{1}{4\pi \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}}.
\]

Now that it’s known, the Green’s function for the ball can be determined by the method of images. A convocation of point charges in infinite space will be arranged so that the boundary condition, \( G = 0 \) along \( \rho = R \), is satisfied.

For a positive unit charge located at \((x_0, y_0, z_0)\) inside the ball, place a charge \( Q^* \) at \((x^*_0, y^*_0, z^*_0)\) outside the ball such that the charges are collinear with the origin. \( x^*_0, y^*_0, z^*_0 \), and \( Q^* \) are all unknown at the moment.
Write the Green's function in the ball (valid for \( x^2 + y^2 + z^2 < R^2 \)).

\[
G(x, y, z; x_0, y_0, z_0) = +g(x, y, z; x_0, y_0, z_0) + Q^*g(x, y, z; x_0^*, y_0^*, z_0^*)
\]

\[
= -\frac{1}{4\pi \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}} - \frac{Q^*}{4\pi \sqrt{(x-x_0^*)^2 + (y-y_0^*)^2 + (z-z_0^*)^2}}
\]

\[
= -\frac{1}{4\pi} \left[ \frac{1}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}} + \frac{Q^*}{\sqrt{(x-x_0^*)^2 + (y-y_0^*)^2 + (z-z_0^*)^2}} \right]
\]

\[
= -\frac{1}{4\pi} \left[ \frac{1}{\sqrt{\rho^2 + \rho_0^2 - 2|x|\rho_0 \cos \alpha}} + \frac{Q^*}{\sqrt{\rho^2 + \rho_0^2 - 2|x|\rho_0 \cos \alpha}} \right]
\]

\[
= -\frac{1}{4\pi} \left[ \frac{1}{\rho \sqrt{1 + \frac{\rho_0^2}{\rho^2} - \frac{2\rho_0 \rho \cos \alpha}{\rho^2}} + \frac{Q^*}{\rho_0 \sqrt{1 + \frac{\rho_0^2}{\rho^2} - \frac{2\rho \rho_0 \cos \alpha}{\rho^2}}} \right]
\]

(2)

\[
G(\rho, \phi, \theta; \rho_0, \phi_0, \theta_0) = -\frac{1}{4\pi} \left[ \frac{1}{\sqrt{\rho^2 + \rho_0^2 - 2\rho \rho_0 \cos \alpha}} + \frac{Q^*}{\sqrt{\rho^2 + \rho_0^2 - 2\rho \rho_0 \cos \alpha}} \right]
\]

Here \( \alpha \) represents the angle between \( x \), the position vector of the point we're interested in knowing the potential at, and \( x_0 \), the position vector of the positive unit charge. Because this positive unit charge and the image charge \( Q^* \) are collinear, \( \alpha \) is also the angle between \( x \) and \( x_0^* \), the position vector of the image. The potential at \( \rho = R \) is zero.

\[
G(R, \phi, \theta; \rho_0, \phi_0, \theta_0) = -\frac{1}{4\pi} \left[ \frac{1}{R \sqrt{1 + \frac{\rho_0^2}{R^2} - \frac{2ho_0 R \cos \alpha}{R^2}}} + \frac{Q^*}{R_0 \sqrt{1 + \frac{\rho_0^2}{R^2} - \frac{2\rho_0 R \cos \alpha}{R^2}}} \right] = 0
\]

In order for the quantity in parentheses to vanish, set

\[
\frac{\rho_0}{R} = \frac{R}{\rho_0} \quad \text{and} \quad \frac{1}{\rho_0} + \frac{Q^*}{\rho_0^2} = 0,
\]

which means

\[
\rho_0^* = \frac{R^2}{\rho_0} \quad \text{and} \quad Q^* = -\frac{\rho_0^*}{R}
\]

\[
Q^* = -\frac{R}{\rho_0}
\]

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Now write equation (2) in terms of spherical coordinates and then substitute these formulas for \( \rho_0^* \) and \( Q^* \).

\[
G(x, y, z; x_0, y_0, z_0) = -\frac{1}{4\pi} \left[ \frac{1}{\sqrt{(x^2 + y^2 + z^2)} + (x_0^2 + y_0^2 + z_0^2)} \right.
\]
\[
\left. + \frac{Q^*}{\sqrt{(x^2 + y^2 + z^2)} + (x_0^2 + y_0^2 + z_0^2)} \right] \cdot \frac{1}{2} \cdot \left( x x_0 + y y_0 + z z_0 \right)
\]

\[
G(\rho, \phi, \theta; \rho_0, \phi_0, \theta_0) = -\frac{1}{4\pi} \left\{ \frac{1}{\sqrt{\rho^2 + \rho_0^2} - 2\rho \rho_0 (\cos \phi \cos \phi_0 \sin \theta \sin \theta_0 + \sin \phi \sin \phi_0 \sin \theta \sin \theta_0 + \cos \theta \cos \theta_0)} \right.
\]
\[
\left. + \frac{Q^*}{\sqrt{\rho^2 + \rho_0^2} - 2\rho \rho_0^* (\cos \phi \cos \phi_0 \sin \theta \sin \theta_0 + \sin \phi \sin \phi_0 \sin \theta \sin \theta_0 + \cos \theta \cos \theta_0)} \right\} \cdot \frac{1}{2} \cdot \left( \rho \rho_0 \cos \phi \cos \phi_0 \sin \theta \sin \theta_0 + \sin \phi \sin \phi_0 \sin \theta \sin \theta_0 + \cos \theta \cos \theta_0 \right)
\]

- \[
= -\frac{1}{4\pi} \left[ \frac{1}{\sqrt{\rho^2 + \rho_0^2} - 2\rho \rho_0 (\cos (\phi - \phi_0) \sin \theta \sin \theta_0 + \cos \theta \cos \theta_0)} \right.
\]
\[
\left. + \frac{Q^*}{\sqrt{\rho^2 + \rho_0^2} - 2\rho \rho_0^* (\cos (\phi - \phi_0) \sin \theta \sin \theta_0 + \cos \theta \cos \theta_0)} \right] \cdot \frac{1}{2} \cdot \left( \rho \rho_0 \cos (\phi - \phi_0) \sin \theta \sin \theta_0 + \cos \theta \cos \theta_0 \right)
\]

- \[
= -\frac{1}{4\pi} \left[ \frac{1}{\sqrt{\rho^2 + \rho_0^2} - 2\rho \rho_0 (\cos (\phi - \phi_0) \sin \theta \sin \theta_0 + \cos \theta \cos \theta_0)} \right.
\]
\[
\left. + \frac{\frac{R}{\rho_0}}{\sqrt{\rho^2 + \rho_0^2} - 2\rho \rho_0 R^2 (\cos (\phi - \phi_0) \sin \theta \sin \theta_0 + \cos \theta \cos \theta_0)} \right] \cdot \frac{1}{2} \cdot \left( \rho \rho_0 \cos (\phi - \phi_0) \sin \theta \sin \theta_0 + \cos \theta \cos \theta_0 \right)
\]

- \[
= -\frac{1}{4\pi} \left[ \frac{1}{\sqrt{\rho^2 + \rho_0^2} - 2\rho \rho_0 (\cos (\phi - \phi_0) \sin \theta \sin \theta_0 + \cos \theta \cos \theta_0)} \right.
\]
\[
\left. + \frac{R}{\sqrt{\rho^2 + \rho_0^2} - 2\rho \rho_0 R^2 (\cos (\phi - \phi_0) \sin \theta \sin \theta_0 + \cos \theta \cos \theta_0)} \right] \cdot \frac{1}{2} \cdot \left( \rho \rho_0 \cos (\phi - \phi_0) \sin \theta \sin \theta_0 + \cos \theta \cos \theta_0 \right)
\]
Therefore, the Green’s function inside a ball of radius $R$ in spherical coordinates is

$$G(\rho, \phi, \theta; \rho_0, \phi_0, \theta_0) = \frac{1}{4\pi} \left[ \frac{R}{\sqrt{\rho^2 \rho_0^2 + R^4 - 2R^2 \rho \rho_0 [\cos(\phi - \phi_0) \sin \theta \sin \theta_0 + \cos \theta \cos \theta_0]} - \frac{1}{\sqrt{\rho^2 + \rho_0^2 - 2 \rho \rho_0 [\cos(\phi - \phi_0) \sin \theta \sin \theta_0 + \cos \theta \cos \theta_0]} \right].$$

Observe that if the positive unit charge is placed at the center of the ball, then the potential at any point inside has a much simpler form.

$$\lim_{\rho_0 \to 0} G(\rho, \phi, \theta; \rho_0, \phi_0, \theta_0) = \frac{1}{4\pi} \left( \frac{R}{\sqrt{R^4}} - \frac{1}{\sqrt{\rho^2}} \right) = \frac{1}{4\pi} \left( \frac{1}{R} - \frac{1}{\rho} \right)$$

This is essentially equation (11) in the textbook. Calculate $\partial G / \partial \rho_0$

$$\frac{\partial G}{\partial \rho_0} = \frac{1}{4\pi} \left\{ \frac{R}{2} \left\{ \rho^2 \rho_0^2 + R^4 - 2R^2 \rho \rho_0 [\cos(\phi - \phi_0) \sin \theta \sin \theta_0 + \cos \theta \cos \theta_0] \right\}^{3/2} + \frac{1}{2} \left\{ \rho^2 + \rho_0^2 - 2 \rho \rho_0 [\cos(\phi - \phi_0) \sin \theta \sin \theta_0 + \cos \theta \cos \theta_0] \right\} \right\}^{3/2}$$

and then evaluate it at $\rho_0 = R$.

$$\frac{\partial G}{\partial \rho_0} \bigg|_{\rho_0=R} = \frac{1}{4\pi} \left\{ \frac{R}{2} \left\{ \rho^2 + R^2 - 2R \rho [\cos(\phi - \phi_0) \sin \theta \sin \theta_0 + \cos \theta \cos \theta_0] \right\}^{3/2} + \frac{1}{2} \left\{ \rho^2 + R^2 - 2R \rho [\cos(\phi - \phi_0) \sin \theta \sin \theta_0 + \cos \theta \cos \theta_0] \right\} \right\}^{3/2}$$

$$= \frac{1}{4\pi} \left\{ \rho^2 + R^2 - 2R \rho [\cos(\phi - \phi_0) \sin \theta \sin \theta_0 + \cos \theta \cos \theta_0] \right\}^{3/2}$$

$$= \frac{1}{4\pi R} \left\{ \rho^2 + R^2 - 2R \rho [\cos(\phi - \phi_0) \sin \theta \sin \theta_0 + \cos \theta \cos \theta_0] \right\}^{3/2}$$

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Therefore, the solution to the Poisson equation inside a ball with a prescribed boundary condition at $\rho = R$ is

$$u(\rho, \phi, \theta) = \frac{1}{4\pi} \int_0^\pi \int_0^{2\pi} \int_0^R \frac{R}{\sqrt{\rho^2 \rho_0^2 + R^4 - 2R^2 \rho \rho_0 [\cos(\phi - \phi_0) \sin \theta \sin \theta_0 + \cos \theta \cos \theta_0]}} \left[ f(\rho_0, \phi_0, \theta_0) \rho_0^2 \sin \theta_0 d\rho_0 d\phi_0 d\theta_0 + \frac{R}{\sqrt{\rho^2 + \rho_0^2 - 2\rho \rho_0 [\cos(\phi - \phi_0) \sin \theta \sin \theta_0 + \cos \theta \cos \theta_0]}} \right] {\rho^2 + R^2 - 2R\rho [\cos(\phi - \phi_0) \sin \theta \sin \theta_0 + \cos \theta \cos \theta_0]}^{3/2} F(\phi_0, \theta_0) \sin \theta_0 d\phi_0 d\theta_0.$$ 

If $f = 0$, then the solution reduces to

$$u(\rho, \phi, \theta) = \frac{R}{4\pi} \int_0^\pi \int_0^{2\pi} \frac{R^2 - \rho^2}{\{\rho^2 + R^2 - 2R\rho [\cos(\phi - \phi_0) \sin \theta \sin \theta_0 + \cos \theta \cos \theta_0]\}^{3/2}} F(\phi_0, \theta_0) \sin \theta_0 d\phi_0 d\theta_0,$$

which is essentially equation (16) in the textbook.