Exercise 11

(a) Verify that (18) is the Green's function for the disk.

(b) Use it to recover the Poisson formula.

Solution

Here the aim is to solve the Poisson equation in a disk with radius R that is subject to a boundary condition.

$$\begin{aligned} &\Delta u = f(r,\theta), \quad r < R,\ 0 < \theta < 2\pi \\ &u(R,\theta) = F(\theta) \end{aligned}$$

A Green's function representation for the solution can be obtained from Green's second identity in two dimensions,

$$\iint_D (u\Delta v - v\Delta u)\, dA = \int_{\text{bdy } D} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) ds,$$

which holds for any two functions, u and v, over any domain and its boundary. Let v be the Green's function: $v = G = G(r,\theta;r_0,\theta_0)$.

$$\iint_D (u\Delta G - G\Delta u)\, dA = \int_{\text{bdy } D} \left(u\frac{\partial G}{\partial n} - G\frac{\partial u}{\partial n}\right) ds \tag{1}$$

If we require it to satisfy

$$\begin{aligned} &\Delta G = \delta(r-r_0)\delta(\theta-\theta_0), \qquad r < R,\ 0 < \theta < 2\pi \\ &G = 0 \text{ at } r = R, \end{aligned}$$

where (r_0,θ_0) is a point in the disk, then equation (1) becomes

$$\iint_D [u(r,\theta)\delta(r-r_0)\delta(\theta-\theta_0) - G(r,\theta;r_0,\theta_0)f(r,\theta)]\, dA = \int_{\text{bdy } D} \left(u\frac{\partial G}{\partial n} - (0)\frac{\partial u}{\partial n}\right) ds.$$

Since the domain is a disk centered at the origin, the outward unit normal vector is $\hat{\mathbf{n}} = \hat{\mathbf{r}}$, which means the normal derivative is $\partial/\partial n = \partial/\partial r$.

$$\iint_D u(r,\theta)\delta(r-r_0)\delta(\theta-\theta_0)\, dA - \iint_D G(r,\theta;r_0,\theta_0)f(r,\theta)\, dA = \int_{\text{bdy } D} u\frac{\partial G}{\partial r}\, ds$$

The integral involving the delta functions is $u(r_0,\theta_0)$.

$$\begin{aligned} u(r_0,\theta_0) - \int_0^{2\pi}\int_0^R G(r,\theta;r_0,\theta_0)f(r,\theta) r\, dr\, d\theta &= \int_0^{2\pi} u(R,\theta)\frac{\partial G}{\partial r}\bigg|_{r=R} (R\, d\theta) \\ &= R\int_0^{2\pi} F(\theta)\frac{\partial G}{\partial r}\bigg|_{r=R} d\theta \end{aligned}$$

Solve for u.

$$u(r_0,\theta_0) = \int_0^{2\pi}\int_0^R G(r,\theta;r_0,\theta_0)f(r,\theta)r\,dr\,d\theta + R\int_0^{2\pi} F(\theta)\left.\frac{\partial G}{\partial r}\right|_{r=R} d\theta$$

Switch the roles of r_0 and θ_0 with those of r and θ, respectively.

$$u(r,\theta) = \int_0^{2\pi}\int_0^R G(r_0,\theta_0;r,\theta)f(r_0,\theta_0)r_0\,dr_0\,d\theta_0 + R\int_0^{2\pi} F(\theta_0)\left.\frac{\partial G}{\partial r_0}\right|_{r_0=R} d\theta_0$$

Therefore, using the fact that the Green's function is symmetric,

$$u(r,\theta) = \int_0^{2\pi}\int_0^R G(r,\theta;r_0,\theta_0)f(r_0,\theta_0)r_0\,dr_0\,d\theta_0 + R\int_0^{2\pi} F(\theta_0)\left.\frac{\partial G}{\partial r_0}\right|_{r_0=R} d\theta_0.$$

The solution for Poisson's equation is known, then, if the Green's function in the disk can be determined. Begin by finding the Green's function in the whole plane (no boundaries).

$$\Delta g = \delta(x-x_0)\delta(y-y_0), \qquad -\infty < x < \infty,\ -\infty < y < \infty$$

g can be interpreted as the electrostatic potential, and $\delta(x-x_0)\delta(y-y_0)$ can be interpreted as the charge density for a unit charge located at (x_0,y_0). Since there are no boundaries, g is expected to vary solely as a function of the radial distance from (x_0,y_0): $g = g(\mathscr{r})$, where $\mathscr{r} = \sqrt{(x-x_0)^2+(y-y_0)^2}$. Integrate both sides over a disk centered at (x_0,y_0) with radius $\mathscr{r}$.

$$\iint\limits_{(x-x_0)^2+(y-y_0)^2\le \mathscr{r}^2} \Delta g\,dA = \iint\limits_{(x-x_0)^2+(y-y_0)^2\le \mathscr{r}^2} \delta(x-x_0)\delta(y-y_0)\,dA$$

Since the disk contains (x_0,y_0), the right side is 1. Write the Laplacian operator as $\Delta = \nabla^2$

$$\iint\limits_{(x-x_0)^2+(y-y_0)^2\le \mathscr{r}^2} \nabla^2 g\,dA = 1$$

$$\iint\limits_{(x-x_0)^2+(y-y_0)^2\le \mathscr{r}^2} \nabla\cdot\nabla g\,dA = 1$$

and apply the two-dimensional divergence theorem.

$$\int\limits_{(x-x_0)^2+(y-y_0)^2= \mathscr{r}^2} \nabla g\cdot\hat{\boldsymbol{\mathscr{r}}}\,ds = 1$$

Here $\hat{\boldsymbol{\mathscr{r}}}$ is the unit vector normal to this disk at every point on the boundary.

$$\int\limits_{(x-x_0)^2+(y-y_0)^2= \mathscr{r}^2} \frac{dg}{d\mathscr{r}}\,ds = 1$$

Because g only depends on $\mathscr{r}$, its derivative is constant on the disk's boundary.

$$\frac{dg}{d\mathscr{r}}\int\limits_{(x-x_0)^2+(y-y_0)^2= \mathscr{r}^2} ds = 1$$

This line integral is just the disk's circumference.

$$\frac{dg}{d\mathscr{r}}(2\pi\mathscr{r}) = 1$$

Divide both sides by $2\pi\mathscr{r}$.

$$\frac{dg}{d\mathscr{r}} = \frac{1}{2\pi\mathscr{r}}$$

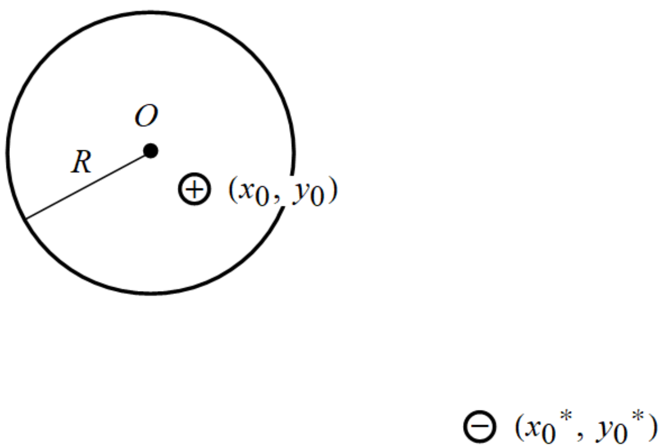
Integrate both sides with respect to $\mathscr{r}$.

$$g(\mathscr{r}) = \frac{1}{2\pi}\ln\mathscr{r}$$

The Green's function for the whole plane is then

$$g(x,y;x_0,y_0) = \frac{1}{2\pi}\ln\sqrt{(x-x_0)^2+(y-y_0)^2}.$$

Now that it's known, the Green's function for the disk can be determined by the method of images. A convocation of point charges in the whole plane will be arranged so that the boundary condition, $G = 0$ along $r = R$, is satisfied.



For a positive unit charge located at (x_0, y_0) inside the disk, place a negative unit charge at (x_0^*, y_0^*) outside the disk so that the potential is zero on the disk's boundary. For the moment, x_0^* and y_0^* are unknown, but because the charges are collinear with the origin, the position vector $\mathbf{x}_0^* = \langle x_0^*, y_0^*\rangle$ is some constant multiple of $\mathbf{x}_0 = \langle x_0, y_0\rangle$: $\mathbf{x}_0^* = C_1\mathbf{x}_0 = \langle C_1x_0, C_1y_0\rangle$. The Green's function in the disk (valid for $x^2 + y^2 < R^2$) will now be written.

$$\begin{aligned} G(x,y;x_0,y_0) &= +g(x,y;x_0,y_0) - g(x,y;x_0^*,y_0^*) \\ &= \frac{1}{2\pi}\ln\sqrt{(x-x_0)^2+(y-y_0)^2} - \frac{1}{2\pi}\ln\sqrt{(x-x_0^*)^2+(y-y_0^*)^2} \\ &= \frac{1}{2\pi}\ln\frac{\sqrt{(x-x_0)^2+(y-y_0)^2}}{\sqrt{(x-x_0^*)^2+(y-y_0^*)^2}} \\ &= \frac{1}{4\pi}\ln\frac{(x-x_0)^2+(y-y_0)^2}{(x-x_0^*)^2+(y-y_0^*)^2} \end{aligned}$$

Since the Laplacian operator acts on G in the PDE, any constant can be added to this result.

$$G(x,y;x_0,y_0) = \frac{1}{4\pi}\ln\frac{(x-x_0)^2+(y-y_0)^2}{(x-x_0^*)^2+(y-y_0^*)^2} + C_2$$

In order for G to vanish on the disk's boundary, the logarithm argument must be a constant there.

$$\frac{(x-x_0)^2+(y-y_0)^2}{(x-x_0^*)^2+(y-y_0^*)^2} = C_3 \quad \text{at } r = R \tag{2}$$

Apply the boundary condition, $G = 0$ at $r = R$, to determine C_2.

$$\frac{1}{4\pi}\ln C_3 + C_2 = 0 \quad \rightarrow \quad C_2 = -\frac{1}{4\pi}\ln C_3$$

Consequently, the Green's function becomes

$$\begin{aligned} G(x,y;x_0,y_0) &= \frac{1}{4\pi}\ln\frac{(x-x_0)^2+(y-y_0)^2}{(x-x_0^*)^2+(y-y_0^*)^2} - \frac{1}{4\pi}\ln C_3 \\ &= \frac{1}{4\pi}\ln\left[\frac{(x-x_0)^2+(y-y_0)^2}{(x-x_0^*)^2+(y-y_0^*)^2}\cdot\frac{1}{C_3}\right] \\ &= \frac{1}{4\pi}\ln\left[\frac{(x-x_0)^2+(y-y_0)^2}{(x-C_1x_0)^2+(y-C_1y_0)^2}\cdot\frac{1}{C_3}\right]. \end{aligned}$$

Only C_1 and C_3 are left to determine now. Rewrite equation (2).

$$\begin{aligned} \text{At } r = R, \qquad C_3 &= \frac{(x-x_0)^2+(y-y_0)^2}{(x-x_0^*)^2+(y-y_0^*)^2} \\ &= \frac{(x-x_0)^2+(y-y_0)^2}{(x-C_1x_0)^2+(y-C_1y_0)^2} \\ &= \frac{x^2-2xx_0+x_0^2+y^2-2yy_0+y_0^2}{x^2-2C_1xx_0+C_1^2x_0^2+y^2-2C_1yy_0+C_1^2y_0^2} \\ &= \frac{r^2+r_0^2-2(xx_0+yy_0)}{r^2+C_1^2r_0^2-2C_1(xx_0+yy_0)} \\ &= \frac{r^2+r_0^2-2(\mathbf{x}\cdot\mathbf{x}_0)}{r^2+C_1^2r_0^2-2C_1(\mathbf{x}\cdot\mathbf{x}_0)} \\ &= \frac{r^2+r_0^2-2|\mathbf{x}||\mathbf{x}_0|\cos\alpha}{r^2+C_1^2r_0^2-2C_1|\mathbf{x}||\mathbf{x}_0|\cos\alpha} \\ &= \frac{r^2+r_0^2-2rr_0\cos\alpha}{r^2+C_1^2r_0^2-2C_1rr_0\cos\alpha} \\ &= \frac{R^2+r_0^2-2Rr_0\cos\alpha}{R^2+C_1^2r_0^2-2C_1Rr_0\cos\alpha} \end{aligned}$$

Here α is the angle between $\mathbf{x}$ and $\mathbf{x}_0$. This equation must hold for any value of α, so set α to any two values to obtain a system of equations for C_1 and C_3.

$$\begin{aligned} \alpha = 0: &\qquad \frac{R^2+r_0^2-2Rr_0}{R^2+C_1^2r_0^2-2C_1Rr_0} = C_3 \\ \alpha = \frac{\pi}{2}: &\qquad \frac{R^2+r_0^2}{R^2+C_1^2r_0^2} = C_3 \end{aligned}$$

Solving this system yields

$$C_1 = \frac{R^2}{r_0^2} \quad \text{and} \quad C_3 = \frac{r_0^2}{R^2}.$$

Substitute these values into the Green's function.

$$G(x,y;x_0,y_0) = \frac{1}{4\pi}\ln\left[\frac{(x-x_0)^2+(y-y_0)^2}{\left(x-\frac{R^2}{r_0^2}x_0\right)^2+\left(y-\frac{R^2}{r_0^2}y_0\right)^2}\cdot\frac{R^2}{r_0^2}\right]$$

In terms of Cartesian coordinates, it is

$$\begin{aligned} G(x,y;x_0,y_0) &= \frac{1}{4\pi}\ln\left[\frac{(x-x_0)^2+(y-y_0)^2}{\left(x-\frac{R^2}{x_0^2+y_0^2}x_0\right)^2+\left(y-\frac{R^2}{x_0^2+y_0^2}y_0\right)^2}\cdot\frac{R^2}{x_0^2+y_0^2}\right] \\ &= \frac{1}{4\pi}\ln\frac{R^2[(x^2+y^2)+(x_0^2+y_0^2)-2(xx_0+yy_0)]}{(x^2+y^2)(x_0^2+y_0^2)-2R^2(xx_0+yy_0)+R^4}, \end{aligned}$$

and in terms of polar coordinates, it is

$$\begin{aligned} G(r,\theta;r_0,\theta_0) &= \frac{1}{4\pi}\ln\frac{R^2\{r^2+r_0^2-2[(r\cos\theta)(r_0\cos\theta_0)+(r\sin\theta)(r_0\sin\theta_0)]\}}{r^2r_0^2-2R^2[(r\cos\theta)(r_0\cos\theta_0)+(r\sin\theta)(r_0\sin\theta_0)]+R^4} \\ &= \frac{1}{4\pi}\ln\frac{R^2[r^2+r_0^2-2(rr_0\cos\theta\cos\theta_0+rr_0\sin\theta\sin\theta_0)]}{r^2r_0^2-2R^2(rr_0\cos\theta\cos\theta_0+rr_0\sin\theta\sin\theta_0)+R^4} \\ &= \frac{1}{4\pi}\ln\frac{R^2[r^2+r_0^2-2rr_0(\cos\theta\cos\theta_0+\sin\theta\sin\theta_0)]}{r^2r_0^2-2R^2rr_0(\cos\theta\cos\theta_0+\sin\theta\sin\theta_0)+R^4} \\ &= \frac{1}{4\pi}\ln\frac{R^2[r^2+r_0^2-2rr_0\cos(\theta-\theta_0)]}{r^2r_0^2-2R^2rr_0\cos(\theta-\theta_0)+R^4}. \end{aligned}$$

Calculate $\partial G/\partial r_0$

$$\begin{aligned}\frac{\partial G}{\partial r_0} &= \frac{1}{4\pi}\frac{r^2r_0^2 - 2R^2rr_0\cos(\theta-\theta_0) + R^4}{R^2[r^2+r_0^2-2rr_0\cos(\theta-\theta_0)]}\frac{\partial}{\partial r_0}\left\{\frac{R^2[r^2+r_0^2-2rr_0\cos(\theta-\theta_0)]}{r^2r_0^2 - 2R^2rr_0\cos(\theta-\theta_0) + R^4}\right\} \\ &= \frac{1}{4\pi}\frac{r^2r_0^2 - 2R^2rr_0\cos(\theta-\theta_0) + R^4}{R^2[r^2+r_0^2-2rr_0\cos(\theta-\theta_0)]} \\ &\quad\times\left\{\frac{R^2[2r_0 - 2r\cos(\theta-\theta_0)][r^2r_0^2 - 2R^2rr_0\cos(\theta-\theta_0) + R^4] - [2r^2r_0 - 2R^2r\cos(\theta-\theta_0)]R^2[r^2+r_0^2-2rr_0\cos(\theta-\theta_0)]}{[r^2r_0^2 - 2R^2rr_0\cos(\theta-\theta_0) + R^4]^2}\right\}\end{aligned}$$

and then evaluate it at $r_0 = R$.

$$\begin{aligned}\left.\frac{\partial G}{\partial r_0}\right|_{r_0=R} &= \frac{1}{4\pi}(1)\left\{\frac{R^2[2R - 2r\cos(\theta-\theta_0)][r^2R^2 - 2R^3r\cos(\theta-\theta_0) + R^4] - [2r^2R - 2R^2r\cos(\theta-\theta_0)]R^2[r^2+R^2-2rR\cos(\theta-\theta_0)]}{[r^2R^2 - 2R^3r\cos(\theta-\theta_0) + R^4]^2}\right\} \\ &= \frac{1}{4\pi}\left\{\frac{R^2[2R - 2r\cos(\theta-\theta_0)] - [2r^2R - 2R^2r\cos(\theta-\theta_0)]}{r^2R^2 - 2R^3r\cos(\theta-\theta_0) + R^4}\right\} \\ &= \frac{1}{4\pi}\left[\frac{2R^3 - 2r^2R}{r^2R^2 - 2R^3r\cos(\theta-\theta_0) + R^4}\right] \\ &= \frac{1}{2\pi R}\left[\frac{R^2 - r^2}{r^2 - 2Rr\cos(\theta-\theta_0) + R^2}\right]\end{aligned}$$

Therefore, the solution to the Poisson equation in a disk of radius R with a boundary condition is

$$\begin{aligned}u(r,\theta) &= \int_0^{2\pi}\int_0^R G(r,\theta;r_0,\theta_0)f(r_0,\theta_0)r_0\,dr_0\,d\theta_0 + R\int_0^{2\pi}F(\theta_0)\left.\frac{\partial G}{\partial r_0}\right|_{r_0=R}d\theta_0 \\ &= \frac{1}{4\pi}\int_0^{2\pi}\int_0^R r_0 f(r_0,\theta_0)\ln\frac{R^2[r^2+r_0^2-2rr_0\cos(\theta-\theta_0)]}{r^2r_0^2 - 2R^2rr_0\cos(\theta-\theta_0) + R^4}\,dr_0\,d\theta_0 + \frac{1}{2\pi}\int_0^{2\pi}F(\theta_0)\frac{R^2-r^2}{r^2 - 2Rr\cos(\theta-\theta_0) + R^2}\,d\theta_0.\end{aligned}$$

If $f = 0$, then it reduces to

$$u(r,\theta) = \frac{R^2-r^2}{2\pi}\int_0^{2\pi}\frac{F(\theta_0)}{r^2 - 2Rr\cos(\theta-\theta_0) + R^2}\,d\theta_0,$$

which is Poisson's formula.