Exercise 13

Find the Green’s function for the half-ball $D = \{ x^2 + y^2 + z^2 < a^2, \ z > 0 \}$. (Hint: The easiest method is to use the solution for the whole ball and reflect it across the plane.)

Solution

The aim here is to solve the Poisson equation inside a half-ball with radius $R$ that is subject to boundary conditions on its spherical and planar surfaces. Use a spherical coordinate system $(\rho, \phi, \theta)$ in which $\theta$ is the angle from the polar axis.

$$\Delta u = f(\rho, \phi, \theta), \quad \rho < R, \ 0 < \phi < 2\pi, \ 0 < \theta < \frac{\pi}{2}$$

$$u(R, \phi, \theta) = F(\phi, \theta)$$

$$u \left( \rho, \phi, \frac{\pi}{2} \right) = H(\rho, \phi)$$

A Green’s function representation for the solution can be obtained from Green’s second identity,

$$\iint_D (u \Delta v - v \Delta u) \, dV = \iint_{\partial D} \left( u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) \, dS,$$

which holds for any two functions, $u$ and $v$, over any domain and its boundary. Let $v$ be the Green’s function: $v = G = G(\rho, \phi, \theta; \rho_0, \phi_0, \theta_0)$.

$$\iint_D (uG - G\Delta u) \, dV = \iint_{\partial D} \left( u \frac{\partial G}{\partial n} - G \frac{\partial u}{\partial n} \right) \, dS$$  \hfill (1)

If we require it to satisfy

$$\Delta G = \delta(\rho - \rho_0)\delta(\phi - \phi_0)\delta(\theta - \theta_0), \quad \rho < R, \ 0 < \phi < 2\pi, \ 0 < \theta < \frac{\pi}{2}$$

$$G = 0 \text{ on } \partial D,$$

where $(\rho_0, \phi_0, \theta_0)$ is a point in the half-ball, then equation (1) becomes

$$\iiint_D [u(\rho, \phi, \theta)\delta(\rho - \rho_0)\delta(\phi - \phi_0)\delta(\theta - \theta_0) - G(\rho, \phi, \theta; \rho_0, \phi_0, \theta_0)f(\rho, \phi, \theta)] \, dV$$

$$= \iint_{\partial D} \left( u \frac{\partial G}{\partial n} - (0) \frac{\partial u}{\partial n} \right) \, dS.$$  

Since the domain is a half-ball centered at the origin, there are two boundaries to consider. There’s the spherical boundary $\rho = R$, where the outward unit normal vector is $\hat{n} = \hat{\rho}$ and the normal derivative is $\partial/\partial n = \partial/\partial \rho$. There’s also the planar boundary $z = 0$, where the outward unit normal vector is $\hat{n} = -\hat{z}$ and the normal derivative is $\partial/\partial n = -\partial/\partial z$.

$$\iiint_D u(\rho, \phi, \theta)\delta(\rho - \rho_0)\delta(\phi - \phi_0)\delta(\theta - \theta_0) \, dV - \iiint_D G(\rho, \phi, \theta; \rho_0, \phi_0, \theta_0)f(\rho, \phi, \theta) \, dV = \iint_{\partial D} \frac{\partial G}{\partial n} \, dS.$$  

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The integral involving the delta functions is \( u(\rho_0, \phi_0, \theta_0) \).

\[
\begin{align*}
\int_0^{\pi/2} \int_0^{2\pi} \int_0^R \frac{\partial G}{\partial \rho} \bigg|_{\rho=R} R^2 \sin \theta \, d\phi \, d\theta + \int_0^{2\pi} \int_0^R \frac{\partial G}{\partial \theta} \bigg|_{\theta=\pi/2} \left( -\frac{\partial G}{\partial z} \right) \bigg|_{z=0} \rho \, d\rho \, d\phi \\
= \int_0^{\pi/2} \int_0^{2\pi} \int_0^R \frac{\partial G}{\partial \rho} \bigg|_{\rho=R} R^2 \sin \theta \, d\phi \, d\theta + \int_0^{2\pi} \int_0^R \frac{\partial G}{\partial \theta} \bigg|_{\theta=\pi/2} \left( -\frac{\partial G}{\partial z} \right) \bigg|_{z=0} \rho \, d\rho \, d\phi
\end{align*}
\]

Solve for \( u \).

\[
\begin{align*}
u(\rho_0, \phi_0, \theta_0) &= \int_0^{\pi/2} \int_0^{2\pi} \int_0^R \frac{\partial G}{\partial \rho} \bigg|_{\rho=R} R^2 \sin \theta \, d\phi \, d\theta + \int_0^{2\pi} \int_0^R \frac{\partial G}{\partial \theta} \bigg|_{\theta=\pi/2} \left( -\frac{\partial G}{\partial z} \right) \bigg|_{z=0} \rho \, d\rho \, d\phi \\
&= \int_0^{\pi/2} \int_0^{2\pi} \int_0^R \frac{\partial G}{\partial \rho} \bigg|_{\rho=R} R^2 \sin \theta \, d\phi \, d\theta + \int_0^{2\pi} \int_0^R \frac{\partial G}{\partial \theta} \bigg|_{\theta=\pi/2} \left( -\frac{\partial G}{\partial z} \right) \bigg|_{z=0} \rho \, d\rho \, d\phi
\end{align*}
\]

In spherical coordinates, \( x = \rho \cos \phi \sin \theta \) and \( y = \rho \sin \phi \sin \theta \) and \( z = \rho \cos \theta \). Solve these three equations for \( \rho \), \( \phi \), and \( \theta \).

\[
\begin{align*}
\rho &= \sqrt{x^2 + y^2 + z^2} \\
\phi &= \tan^{-1} \left( \frac{y}{x} \right) \\
\theta &= \tan^{-1} \left( \frac{\sqrt{x^2 + y^2}}{z} \right)
\end{align*}
\]

Use the chain rule to write \( \partial G/\partial z \) in spherical coordinates.

\[
\begin{align*}
\frac{\partial G}{\partial z} &= \frac{\partial G}{\partial \rho} \frac{\partial \rho}{\partial z} + \frac{\partial G}{\partial \phi} \frac{\partial \phi}{\partial z} + \frac{\partial G}{\partial \theta} \frac{\partial \theta}{\partial z} \\
&= \frac{\partial G}{\partial \rho} \left( \frac{1}{2} (x^2 + y^2 + z^2)^{-1/2} (2z) \right) + \frac{\partial G}{\partial \phi} (0) + \frac{\partial G}{\partial \theta} \left[ \frac{1}{1 + \left( \frac{\sqrt{x^2 + y^2}}{z} \right)^2} \left( -\frac{\sqrt{x^2 + y^2}}{z^2} \right) \right] \\
&= \frac{\partial G}{\partial \rho} \left( \frac{z}{\sqrt{x^2 + y^2 + z^2}} \right) - \frac{\partial G}{\partial \theta} \left( \frac{\sqrt{x^2 + y^2}}{x^2 + y^2 + z^2} \right) \\
&= \frac{\partial G}{\partial \rho} (\cos \theta) - \frac{\partial G}{\partial \theta} \left( \frac{\sin \theta}{\rho} \right)
\end{align*}
\]

\( z = 0 \) corresponds to \( \theta = \pi/2 \), so

\[
\left. \frac{\partial G}{\partial z} \right|_{z=0} = \left[ \frac{\partial G}{\partial \rho} (\cos \theta) - \frac{\partial G}{\partial \theta} \left( \frac{\sin \theta}{\rho} \right) \right]_{\theta=\pi/2} = -\frac{1}{\rho} \frac{\partial G}{\partial \theta} \bigg|_{\theta=\pi/2}.
\]
As a result,

\[
u(\rho_0, \phi_0, \theta_0) = \int_0^{\pi/2} \int_0^{2\pi} \int_0^R G(\rho, \phi, \theta; \rho_0, \phi_0, \theta_0) f(\rho, \phi, \theta) \rho^2 \sin \theta \, d\rho \, d\phi \, d\theta \\
+ R^2 \int_0^{\pi/2} \int_0^{2\pi} F(\phi, \theta) \frac{\partial G}{\partial \rho} \bigg|_{\rho=R} \sin \theta \, d\phi \, d\theta \\
- \int_0^{2\pi} \int_0^R H(\rho, \phi) \left( -\frac{1}{\rho} \frac{\partial G}{\partial \theta} \bigg|_{\theta=\pi/2} \right) \rho \, d\rho \, d\phi.
\]

Switch the roles of \(\rho_0, \phi_0,\) and \(\theta_0\) with those of \(\rho, \phi,\) and \(\theta,\) respectively.

\[
u(\rho, \phi, \theta) = \int_0^{\pi/2} \int_0^{2\pi} \int_0^R G(\rho_0, \phi_0, \theta_0; \rho, \phi, \theta) \rho_0^2 \sin \theta_0 \, d\rho_0 \, d\phi_0 \, d\theta_0 \\
+ R^2 \int_0^{\pi/2} \int_0^{2\pi} F(\phi_0, \theta_0) \frac{\partial G}{\partial \rho_0} \bigg|_{\rho_0=R} \sin \theta_0 \, d\phi_0 \, d\theta_0 \\
+ \int_0^{2\pi} \int_0^R H(\rho_0, \phi_0) \frac{\partial G}{\partial \theta_0} \bigg|_{\theta_0=\pi/2} \rho_0 \, d\rho_0 \, d\phi_0.
\]

Therefore, using the fact that the Green’s function is symmetric,

\[
u(\rho, \phi, \theta) = \int_0^{\pi/2} \int_0^{2\pi} \int_0^R G(\rho, \phi, \theta; \rho_0, \phi_0, \theta_0) f(\rho_0, \phi_0, \theta_0) \rho_0^2 \sin \theta_0 \, d\rho_0 \, d\phi_0 \, d\theta_0 \\
+ R^2 \int_0^{\pi/2} \int_0^{2\pi} F(\phi_0, \theta_0) \frac{\partial G}{\partial \rho_0} \bigg|_{\rho_0=R} \sin \theta_0 \, d\phi_0 \, d\theta_0 \\
+ \int_0^{2\pi} \int_0^R H(\rho_0, \phi_0) \frac{\partial G}{\partial \theta_0} \bigg|_{\theta_0=\pi/2} \rho_0 \, d\rho_0 \, d\phi_0.
\]

The solution for Poisson’s equation is known, then, if the Green’s function inside the half-ball can be determined. Begin by finding the Green’s function in infinite space (no boundaries).

\[
\Delta g = \delta(x - x_0) \delta(y - y_0) \delta(z - z_0), \quad -\infty < x, y, z < \infty
\]

\(g\) can be interpreted as the electrostatic potential, and \(\delta(x - x_0) \delta(y - y_0) \delta(z - z_0)\) can be interpreted as the charge density for a unit charge located at \((x_0, y_0, z_0)\). Since there are no boundaries, \(g\) is expected to vary solely as a function of the radial distance from \((x_0, y_0, z_0)\):

\(g = g(\rho),\) where \(\rho = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}\). Integrate both sides over a solid ball centered at \((x_0, y_0, z_0)\) with radius \(\rho\).

\[
\iiint_{\frac{(x-x_0)^2 + (y-y_0)^2}{(z-z_0)^2} \leq \rho^2} \Delta g \, dV = \iiint_{\frac{(x-x_0)^2 + (y-y_0)^2}{(z-z_0)^2} \leq \rho^2} \delta(x - x_0) \delta(y - y_0) \delta(z - z_0) \, dV
\]

Since the ball contains \((x_0, y_0, z_0)\), the right side is 1. Write the Laplacian operator \(\Delta\) as \(\nabla^2\)

\[
\iiint_{\frac{(x-x_0)^2 + (y-y_0)^2}{(z-z_0)^2} \leq \rho^2} \nabla^2 g \, dV = 1
\]

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and apply the divergence theorem.

\[ \iiint (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = r^2 \nabla g \cdot \hat{n} dS = 1 \]

Here \( \hat{n} \) is the unit vector normal to this ball at every point on the boundary.

\[ \iiint (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = r^2 \frac{dg}{dr} dS = 1 \]

Because \( g \) only depends on \( r \), its derivative is constant on the ball’s boundary.

\[ \frac{dg}{dr} \iiint (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = r^2 dS = 1 \]

This surface integral is just the ball’s surface area.

\[ \frac{dg}{dr} (4\pi r^2) = 1 \quad \rightarrow \quad \frac{dg}{dr} = \frac{1}{4\pi r^2} \quad \rightarrow \quad g(r) = -\frac{1}{4\pi r} \]

The infinite-space Green’s function is then

\[ g(x, y, z; x_0, y_0, z_0) = -\frac{1}{4\pi \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}}. \]

Now that it’s known, the Green’s function for the half-ball can be determined by the method of images. A convolution of point charges in infinite space will be arranged so that the boundary conditions, \( G = 0 \) along \( \rho = R \) and \( G = 0 \) along \( z = 0 \), are satisfied.

For a positive unit charge located at \((x_0, y_0, z_0)\) inside the half-ball, place a charge \( Q^* \) at \((x_0^*, y_0^*, z_0^*)\) outside the half-ball such that the charges are collinear with the origin. Then place corresponding charges of opposite polarity at their reflections over the \( z = 0 \) plane. \((x_0^*, y_0^*, -z_0^*)\) and \(-Q^*\) are all unknown at the moment.
Write the Green’s function in the half-ball (valid for \(x^2 + y^2 + z^2 < R^2, z > 0\)).

\[
G(x, y, z; x_0, y_0, z_0) = \frac{1}{4\pi} \frac{1}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}} Q^* - \frac{1}{4\pi} \frac{1}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z+z_0)^2}} Q^* \]

\[
= -\frac{1}{4\pi} \left[ \frac{1}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}} + \frac{Q^*}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z+z_0)^2}} \right]
\]

\[
= -\frac{1}{4\pi} \left[ \frac{1}{\sqrt{(x^2 + y^2 + z^2) + (x_0^2 + y_0^2 + z_0^2) - 2(x_0x + y_0y + zz_0)}} \right. \\
\left. + \frac{Q^*}{\sqrt{(x^2 + y^2 + z^2) + (x_0^2 + y_0^2 + z_0^2) - 2(x_0x + y_0y + zz_0)}} \right]
\]

\[
+ \frac{1}{4\pi} \left[ \frac{1}{\sqrt{(x^2 + y^2 + z^2) + (x_0^2 + y_0^2 + z_0^2) - 2(x_0x + y_0y + zz_0)}} \right. \\
\left. + \frac{Q^*}{\sqrt{(x^2 + y^2 + z^2) + (x_0^2 + y_0^2 + z_0^2) - 2(x_0x + y_0y + zz_0)}} \right]
\]
Change to spherical coordinates and simplify the formula.

\[
G(\rho, \phi, \theta; \rho_0, \phi_0, \theta_0) = -\frac{1}{4\pi} \left[ \frac{1}{\sqrt{\rho^2 + \rho_0^2 - 2[(\rho \cos \phi \sin \theta)(\rho_0 \cos \phi_0 \sin \theta_0) + (\rho \sin \phi \sin \theta)(\rho_0 \sin \phi_0 \sin \theta_0) + (\rho \cos \theta)(\rho_0 \cos \theta_0)]} \right]
\]

\[
+ \frac{1}{4\pi} \left[ \frac{Q^*}{\sqrt{\rho^2 + \rho_0^2 - 2[(\rho \cos \phi \sin \theta)(\rho_0 \cos \phi_0 \sin \theta_0) + (\rho \sin \phi \sin \theta)(\rho_0 \sin \phi_0 \sin \theta_0) - (\rho \cos \theta)(\rho_0 \cos \theta_0)]} \right]
\]

\[
+ \frac{1}{4\pi} \left[ \frac{Q^*}{\sqrt{\rho^2 + \rho_0^2 - 2[(\rho \cos \phi \sin \theta)(\rho_0 \cos \phi_0 \sin \theta_0) + (\rho \sin \phi \sin \theta)(\rho_0 \sin \phi_0 \sin \theta_0) + (\rho \cos \theta)(\rho_0 \cos \theta_0)]} \right]
\]

\[
+ \frac{1}{4\pi} \left[ \frac{Q^*}{\sqrt{\rho^2 + \rho_0^2 - 2[(\rho \cos \phi \sin \theta)(\rho_0 \cos \phi_0 \sin \theta_0) + (\rho \sin \phi \sin \theta)(\rho_0 \sin \phi_0 \sin \theta_0) - (\rho \cos \theta)(\rho_0 \cos \theta_0)]} \right]
\]

\[
+ \frac{1}{4\pi} \left[ \frac{Q^*}{\sqrt{\rho^2 + \rho_0^2 - 2[(\rho \cos \phi \sin \theta)(\rho_0 \cos \phi_0 \sin \theta_0) + (\rho \sin \phi \sin \theta)(\rho_0 \sin \phi_0 \sin \theta_0) - (\rho \cos \theta)(\rho_0 \cos \theta_0)]} \right]
\]

\[
= -\frac{1}{4\pi} \left[ \frac{1}{\sqrt{\rho^2 + \rho_0^2 - 2\rho \rho_0 (\cos \phi \sin \phi_0 \sin \theta_0 \sin \theta + \sin \phi \sin \phi_0 \sin \theta_0 \sin \theta + \cos \theta \cos \theta_0)} \right]
\]

\[
+ \frac{1}{4\pi} \left[ \frac{Q^*}{\sqrt{\rho^2 + \rho_0^2 - 2\rho \rho_0 (\cos \phi \sin \phi_0 \sin \theta_0 \sin \theta + \sin \phi \sin \phi_0 \sin \theta_0 - \cos \theta \cos \theta_0)} \right]
\]

\[
+ \frac{1}{4\pi} \left[ \frac{Q^*}{\sqrt{\rho^2 + \rho_0^2 - 2\rho \rho_0 (\cos \phi \sin \phi_0 \sin \theta_0 \sin \theta + \sin \phi \sin \phi_0 \sin \theta_0 - \cos \theta \cos \theta_0)} \right]
\]

\[
+ \frac{1}{4\pi} \left[ \frac{Q^*}{\sqrt{\rho^2 + \rho_0^2 - 2\rho \rho_0 (\cos \phi \sin \phi_0 \sin \theta_0 \sin \theta + \sin \phi \sin \phi_0 \sin \theta_0 - \cos \theta \cos \theta_0)} \right]
\]

(2)
Bring $\rho$ out of the square root in the terms with 1 in the numerator, and bring $\rho_0^*$ out of the square root in the terms with $Q^*$ in the numerator.

$$G(\rho, \phi, \theta; \rho_0, \phi_0, \theta_0) = \frac{1}{4\pi} \left[ \frac{1}{\rho} \sqrt{1 + \frac{\rho_0^2}{\rho^2} - \frac{2\rho_0}{\rho} [\cos(\phi - \phi_0) \sin \theta \sin \theta_0 + \cos \theta \cos \theta_0]} + \frac{Q^*}{\rho_0} \sqrt{1 + \frac{\rho_0^*}{\rho_0^2} - \frac{2\rho_0^*}{\rho_0} [\cos(\phi - \phi_0) \sin \theta \sin \theta_0 + \cos \theta \cos \theta_0]} \right]$$

$$+ \frac{1}{4\pi} \left[ \frac{1}{\rho} \sqrt{1 + \frac{\rho_0^2}{\rho^2} - \frac{2\rho_0}{\rho} [\cos(\phi - \phi_0) \sin \theta \sin \theta_0 - \cos \theta \cos \theta_0]} + \frac{Q^*}{\rho_0} \sqrt{1 + \frac{\rho_0^*}{\rho_0^2} - \frac{2\rho_0^*}{\rho_0^2} [\cos(\phi - \phi_0) \sin \theta \sin \theta_0 - \cos \theta \cos \theta_0]} \right]$$

The potential at $\rho = R$ is zero.

$$G(R, \phi, \theta; \rho_0, \phi_0, \theta_0) = \frac{1}{4\pi} \left[ \frac{1}{R} \sqrt{1 + \frac{\rho_0^2}{R^2} - \frac{2\rho_0}{R} [\cos(\phi - \phi_0) \sin \theta \sin \theta_0 + \cos \theta \cos \theta_0]} + \frac{Q^*}{\rho_0} \sqrt{1 + \frac{\rho_0^*}{\rho_0^2} - \frac{2\rho_0^*}{\rho_0^2} [\cos(\phi - \phi_0) \sin \theta \sin \theta_0 + \cos \theta \cos \theta_0]} \right]$$

$$+ \frac{1}{4\pi} \left[ \frac{1}{R} \sqrt{1 + \frac{\rho_0^2}{R^2} - \frac{2\rho_0}{R} [\cos(\phi - \phi_0) \sin \theta \sin \theta_0 - \cos \theta \cos \theta_0]} + \frac{Q^*}{\rho_0} \sqrt{1 + \frac{\rho_0^*}{\rho_0^2} - \frac{2\rho_0^*}{\rho_0^2} [\cos(\phi - \phi_0) \sin \theta \sin \theta_0 - \cos \theta \cos \theta_0]} \right] = 0$$

In order for the quantities in square brackets to vanish, set

$$\frac{\rho_0}{R} = \frac{\rho_0^*}{R_0} \quad \text{and} \quad \frac{1}{R} + \frac{Q^*}{\rho_0} = 0,$$

which means

$$\rho_0^* = \frac{R^2}{\rho_0} \quad \text{and} \quad Q^* = -\frac{\rho_0^*}{R}.$$

$$Q^* = -\frac{R}{\rho_0}.$$
Consequently, the formula for the Green’s function in equation (2) becomes

\[
G(\rho, \phi, \theta; \rho_0, \phi_0, \theta_0) = -\frac{1}{4\pi} \left[ \frac{1}{\sqrt{\rho^2 + \rho_0^2 - 2\rho \rho_0 [\cos(\phi - \phi_0) \sin \theta \sin \theta_0 + \cos \theta \cos \theta_0]} - \frac{R}{\rho_0} \sqrt{\rho^2 + \rho_0^2 + R^2 - 2\rho R \rho_0 [\cos(\phi - \phi_0) \sin \theta \sin \theta_0 + \cos \theta \cos \theta_0]} \right] + \frac{1}{4\pi} \left[ \frac{1}{\sqrt{\rho^2 + \rho_0^2 - 2\rho \rho_0 [\cos(\phi - \phi_0) \sin \theta \sin \theta_0 - \cos \theta \cos \theta_0]} - \frac{R}{\rho_0} \sqrt{\rho^2 + \rho_0^2 + R^2 - 2\rho R \rho_0 [\cos(\phi - \phi_0) \sin \theta \sin \theta_0 - \cos \theta \cos \theta_0]} \right]
\]

\[
= -\frac{1}{4\pi} \left\{ \frac{1}{\sqrt{\rho^2 + \rho_0^2 - 2\rho \rho_0 [\cos(\phi - \phi_0) \sin \theta \sin \theta_0 - \cos \theta \cos \theta_0]} - \frac{1}{\sqrt{\rho^2 + \rho_0^2 - 2\rho \rho_0 [\cos(\phi - \phi_0) \sin \theta \sin \theta_0 + \cos \theta \cos \theta_0]} + R \left[ \frac{1}{\sqrt{\rho^2 + \rho_0^2 + R^2 - 2\rho R \rho_0 [\cos(\phi - \phi_0) \sin \theta \sin \theta_0 + \cos \theta \cos \theta_0]} - \frac{1}{\sqrt{\rho^2 + \rho_0^2 + R^2 - 2\rho R \rho_0 [\cos(\phi - \phi_0) \sin \theta \sin \theta_0 - \cos \theta \cos \theta_0]} \right] \right\}
\]

Calculate the derivatives, \(\partial G/\partial \rho_0\) and \(\partial G/\partial \theta_0\), and evaluate them at \(\rho_0 = R\) and \(\theta_0 = \pi/2\), respectively.

\[
\left. \frac{\partial G}{\partial \rho_0} \right|_{\rho_0=R} = \frac{R^2 - \rho^2}{4\pi R} \left\{ \frac{1}{\{R^2 + \rho^2 - 2R \rho \cos(\phi - \phi_0) \sin \theta \sin \theta_0 + \cos \theta \cos \theta_0\}} \right\}^{3/2} - \frac{1}{\{R^2 + \rho^2 - 2R \rho \cos(\phi - \phi_0) \sin \theta \sin \theta_0 - \cos \theta \cos \theta_0\}} \right\}^{3/2}
\]

\[
\left. \frac{\partial G}{\partial \theta_0} \right|_{\theta_0=\pi/2} = \frac{\rho \rho_0 \cos \theta}{2\pi} \left\{ \frac{1}{\{\rho^2 + \rho_0^2 - 2\rho \rho_0 \cos(\phi - \phi_0) \sin \theta\}} \right\}^{3/2} - \frac{R^3}{\{\rho^2 \rho_0^2 + R^4 - 2\rho^2 \rho_0 \cos(\phi - \phi_0) \sin \theta\}} \right\}^{3/2}
\]

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