Exercise 17

(a) Find the Green’s function for the quadrant

\[ Q = \{(x, y) : x > 0, y > 0\} . \]

(Hint: Either use the method of reflection or reduce to the half-plane problem by the transformation in Exercise 15.)

(b) Use your answer in part (a) to solve the Dirichlet problem

\[ u_{xx} + u_{yy} = 0 \text{ in } Q, \quad u(0, y) = g(y) \text{ for } y > 0, \]

\[ u(x, 0) = h(x) \text{ for } x > 0. \]

Solution

The Poisson equation will be solved in the quarter-plane with two prescribed boundary conditions on the x- and y-axes.

\[ \Delta u = f(x, y), \quad x > 0, y > 0 \]

\[ u(0, y) = g(y) \]

\[ u(x, 0) = h(x) \]

A Green’s function representation for the solution can be obtained from Green’s second identity,

\[ \iint_{Q} (u \Delta v - v \Delta u)\,dA = \int_{\partial Q} \left( u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right)\,ds. \]

Let \( v = G = G(x, y; x_0, y_0) \) be the Green’s function.

\[ \iint_{Q} (u \Delta G - G \Delta u)\,dA = \int_{\partial Q} \left( u \frac{\partial G}{\partial n} - G \frac{\partial u}{\partial n} \right)\,ds \]

If we require it to satisfy

\[ \Delta G = \delta(x-x_0)\delta(y-y_0), \quad x > 0, y > 0 \]

\[ G = 0 \text{ on } x = 0, y = 0, \]

where \((x_0, y_0)\) is a point in the first quadrant, then the identity becomes

\[ \iint_{Q} [u(x, y)\delta(x-x_0)\delta(y-y_0) - G(x, y; x_0, y_0)f(x, y)]\,dA = \int_{\partial Q} \left( u \frac{\partial G}{\partial n} - (0) \frac{\partial u}{\partial n} \right)\,ds. \]

The normal derivative of \( G \) can be written as \( \partial G/\partial n = \nabla G \cdot \hat{n} \), where \( \hat{n} \) is the outward unit vector normal to the boundary.

\[ \iint_{Q} u(x, y)\delta(x-x_0)\delta(y-y_0)\,dA - \iint_{Q} G(x, y; x_0, y_0)f(x, y)\,dA = \int_{\partial Q} u\nabla G \cdot \hat{n}\,ds \]

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Since \((x_0, y_0)\) lies in the first quadrant, the integral on the left involving the delta functions is \(u(x_0, y_0)\). Also, integrating around the boundary counterclockwise yields two integrals, one along the \(y\)-axis and one alone the \(x\)-axis.

\[
u(x_0, y_0) = \int_0^\infty \int_0^\infty G(x, y; x_0, y_0) f(x, y) \, dx \, dy - \int_0^\infty \int_0^\infty G(x, y; x_0, y_0) \frac{\partial G}{\partial x} \bigg|_{x=0} \, dy - \int_0^\infty \int_0^\infty G(x, y; x_0, y_0) \frac{\partial G}{\partial y} \bigg|_{y=0} \, dx
\]

Solve this equation for \(u\).

\[
u(x_0, y_0) = \int_0^\infty \int_0^\infty G(x, y; x_0, y_0) f(x, y) \, dx \, dy - \int_0^\infty \int_0^\infty G(x, y; x_0, y_0) \frac{\partial G}{\partial x} \bigg|_{x=0} \, dy - \int_0^\infty \int_0^\infty G(x, y; x_0, y_0) \frac{\partial G}{\partial y} \bigg|_{y=0} \, dx
\]

Switch the roles of \(x_0\) and \(y_0\) with those of \(x\) and \(y\), respectively.

\[
u(x, y) = \int_0^\infty \int_0^\infty G(x, y; x_0, y_0) f(x, y) \, dx \, dy - \int_0^\infty \int_0^\infty G(x, y; x_0, y_0) \frac{\partial G}{\partial x} \bigg|_{x=0} \, dy - \int_0^\infty \int_0^\infty G(x, y; x_0, y_0) \frac{\partial G}{\partial y} \bigg|_{y=0} \, dx
\]

Therefore, using the fact that the Green’s function is symmetric,

\[
u(x, y) = \int_0^\infty \int_0^\infty G(x, y; x_0, y_0) f(x, y) \, dx \, dy - \int_0^\infty \int_0^\infty G(x, y; x_0, y_0) \frac{\partial G}{\partial x} \bigg|_{x=0} \, dy - \int_0^\infty \int_0^\infty G(x, y; x_0, y_0) \frac{\partial G}{\partial y} \bigg|_{y=0} \, dx
\]

The solution for Poisson’s equation is known, then, if the Green’s function in the quarter-plane can be determined. Begin by finding the Green’s function in the whole plane (no boundaries).

\[\Delta g = \delta(x - x_0)\delta(y - y_0), \quad -\infty < x < \infty, \quad -\infty < y < \infty\]

\(g\) can be interpreted as the electrostatic potential, and \(\delta(x - x_0)\delta(y - y_0)\) can be interpreted as the charge density for a unit charge located at \((x_0, y_0)\). Since there are no boundaries, \(g\) is expected to vary solely as a function of the radial distance from \((x_0, y_0)\): \(g = g(\hat{r})\), where \(\hat{r} = \sqrt{(x - x_0)^2 + (y - y_0)^2}\). Integrate both sides over a disk centered at \((x_0, y_0)\) with radius \(\hat{r}\).

\[
\iint_{(x-x_0)^2+(y-y_0)^2 \leq \hat{r}^2} \Delta g \, dA = \iint_{(x-x_0)^2+(y-y_0)^2 \leq \hat{r}^2} \delta(x - x_0)\delta(y - y_0) \, dA
\]
Since the disk contains \((x_0, y_0)\), the right side is 1. Write the Laplacian operator as \(\Delta = \nabla^2\)

\[
\iint_{(x-x_0)^2 + (y-y_0)^2 \leq r^2} \nabla^2 g \, dA = 1
\]

\[
\iint_{(x-x_0)^2 + (y-y_0)^2 \leq r^2} \nabla \cdot \nabla g \, dA = 1
\]

and apply the two-dimensional divergence theorem.

\[
\int_{(x-x_0)^2 + (y-y_0)^2 = r^2} \nabla g \cdot \hat{n} \, ds = 1
\]

Here \(\hat{n}\) is the unit vector normal to this disk at every point on the boundary.

\[
\int_{(x-x_0)^2 + (y-y_0)^2 = r^2} \frac{dg}{d\varepsilon} \, ds = 1
\]

Because \(g\) only depends on \(\varepsilon\), its derivative is constant on the disk’s boundary.

\[
\frac{dg}{d\varepsilon} \int_{(x-x_0)^2 + (y-y_0)^2 = r^2} ds = 1
\]

This line integral is just the disk’s circumference.

\[
\frac{dg}{d\varepsilon} (2\pi \varepsilon) = 1
\]

Divide both sides by \(2\pi \varepsilon\).

\[
\frac{dg}{d\varepsilon} = \frac{1}{2\pi \varepsilon}
\]

Integrate both sides with respect to \(\varepsilon\).

\[
g(\varepsilon) = \frac{1}{2\pi} \ln \varepsilon
\]

The Green’s function for the whole plane is then

\[
g(x, y; x_0, y_0) = \frac{1}{2\pi} \ln \sqrt{(x - x_0)^2 + (y - y_0)^2}.
\]

Now that it’s known, the Green’s function for the upper half-plane can be determined by the method of images. A convocation of point charges in the whole plane will be arranged so that the boundary condition, \(G = 0\) on \(x = 0\) and \(y = 0\), is satisfied.
For a positive unit charge located at \((x_0, y_0)\), two negative unit charges at \((-x_0, y_0)\) and \((x_0, -y_0)\) and one positive unit charge at \((-x_0, -y_0)\) should be placed. This way, the potential due to each positive charge is cancelled by that due to a negative charge at every point on the boundary.

The quarter-plane Green’s function can now be written.

\[
G(x, y; x_0, y_0) = +g(x, y; x_0, y_0) - g(x, y; -x_0, y_0) - g(x, y; x_0, -y_0) + g(x, y; -x_0, -y_0), \quad x > 0, \ y > 0
\]

Since \(g\) is defined over the whole plane, it’s important to note the restriction to \(x > 0, \ y > 0\) for \(G\).

\[
G(x, y; x_0, y_0) = \frac{1}{2\pi} \ln \sqrt{(x-x_0)^2 + (y-y_0)^2} - \frac{1}{2\pi} \ln \sqrt{(x+x_0)^2 + (y-y_0)^2} - \frac{1}{2\pi} \ln \sqrt{(x-x_0)^2 + (y+y_0)^2} + \frac{1}{2\pi} \ln \sqrt{(x+x_0)^2 + (y+y_0)^2}
\]

\[
= \frac{1}{2\pi} \ln \left[ \frac{\sqrt{x-x_0)^2 + (y-y_0)^2} \sqrt{x+x_0)^2 + (y-y_0)^2} \frac{\sqrt{x-x_0)^2 + (y+y_0)^2} \sqrt{x+x_0)^2 + (y+y_0)^2} \right]
\]

\[
= \frac{1}{4\pi} \ln \left[ \frac{(x-x_0)^2 + (y-y_0)^2}{(x+x_0)^2 + (y-y_0)^2} \right] \left[ \frac{(x-x_0)^2 + (y+y_0)^2}{(x+x_0)^2 + (y+y_0)^2} \right]
\]

Calculate \(\frac{\partial G}{\partial x_0}\)

\[
\frac{\partial G}{\partial x_0} = \frac{1}{4\pi} \frac{-2(x-x_0)}{(x-x_0)^2 + (y-y_0)^2} - \frac{1}{4\pi} \frac{2(x+x_0)}{(x+x_0)^2 + (y-y_0)^2} - \frac{1}{4\pi} \frac{-2(x-x_0)}{(x-x_0)^2 + (y+y_0)^2} + \frac{1}{4\pi} \frac{2(x+x_0)}{(x+x_0)^2 + (y+y_0)^2}
\]

and evaluate it at \(x_0 = 0\).

\[
\frac{\partial G}{\partial x_0} \bigg|_{x_0=0} = \frac{1}{4\pi} \frac{-2x}{x^2 + (y-y_0)^2} - \frac{1}{4\pi} \frac{2x}{x^2 + (y-y_0)^2} - \frac{1}{4\pi} \frac{-2x}{x^2 + (y+y_0)^2} + \frac{1}{4\pi} \frac{2x}{x^2 + (y+y_0)^2}
\]

\[
= -\frac{x}{\pi} \left[ \frac{1}{x^2 + (y-y_0)^2} - \frac{1}{x^2 + (y+y_0)^2} \right]
\]

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Calculate $\partial G/\partial y$

$$\frac{\partial G}{\partial y} = \frac{1}{4\pi} \frac{-2(y - y_0)}{(x - x_0)^2 + (y - y_0)^2} - \frac{1}{4\pi} \frac{-2(y - y_0)}{(x + x_0)^2 + (y - y_0)^2} - \frac{1}{4\pi} \frac{2(y + y_0)}{(x - x_0)^2 + (y + y_0)^2} + \frac{1}{4\pi} \frac{2(y + y_0)}{(x + x_0)^2 + (y + y_0)^2}$$

and evaluate it at $y_0 = 0$.

$$\left. \frac{\partial G}{\partial y} \right|_{y_0=0} = \frac{1}{4\pi} \frac{-2y}{(x - x_0)^2 + y^2} - \frac{1}{4\pi} \frac{-2y}{(x + x_0)^2 + y^2} - \frac{1}{4\pi} \frac{2y}{(x - x_0)^2 + y^2} + \frac{1}{4\pi} \frac{2y}{(x + x_0)^2 + y^2}$$

$$= -\frac{y}{\pi} \left[ \frac{1}{(x - x_0)^2 + y^2} - \frac{1}{(x + x_0)^2 + y^2} \right]$$

Therefore,

$$u(x, y) = \frac{1}{4\pi} \int_0^\infty \int_0^\infty f(x_0, y_0) \ln \left[ \frac{(x - x_0)^2 + (y - y_0)^2}{(x + x_0)^2 + (y + y_0)^2} \right] dx_0 dy_0$$

$$+ \frac{x}{\pi} \int_0^\infty g(y_0) \left[ \frac{1}{x^2 + (y - y_0)^2} - \frac{1}{x^2 + (y + y_0)^2} \right] dy_0$$

$$+ \frac{y}{\pi} \int_0^\infty h(x_0) \left[ \frac{1}{(x - x_0)^2 + y^2} - \frac{1}{(x + x_0)^2 + y^2} \right] dx_0.$$