Exercise 18

(a) Find the Green's function for the octant $\mathscr{O} = \{(x, y, z) : x > 0,\ y > 0,\ z > 0\}$. (*Hint:* Use the method of reflection.)

(b) Use your answer in part (a) to solve the Dirichlet problem

$$\begin{cases} u_{xx} + u_{yy} + u_{zz} = 0 \quad \text{in } \mathscr{O} \\ u(0, y, z) = 0,\ u(x, 0, z) = 0, \quad u(x, y, 0) = h(x, y) \quad \text{for } x > 0,\ y > 0,\ z > 0. \end{cases}$$

Solution

The Poisson equation will be solved in the octant $\mathscr{O}$ with three prescribed boundary conditions on the yz-, xz-, and xy-faces.

$$\begin{aligned} \Delta u &= a(x, y, z), \qquad x > 0,\ y > 0,\ z > 0 \\ u(0, y, z) &= b(y, z) \\ u(x, 0, z) &= c(x, z) \\ u(x, y, 0) &= h(x, y) \end{aligned}$$

A Green's function representation for the solution can be obtained from Green's second identity,

$$\iiint_{\mathscr{O}} (u\Delta v - v\Delta u)\, dV = \iint_{\text{bdy } \mathscr{O}} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) dS.$$

Let $v = G = G(x, y, z; x_0, y_0, z_0)$ be the Green's function.

$$\iiint_{\mathscr{O}} (u\Delta G - G\Delta u)\, dV = \iint_{\text{bdy } \mathscr{O}} \left(u\frac{\partial G}{\partial n} - G\frac{\partial u}{\partial n} \right) dS$$

If we require it to satisfy

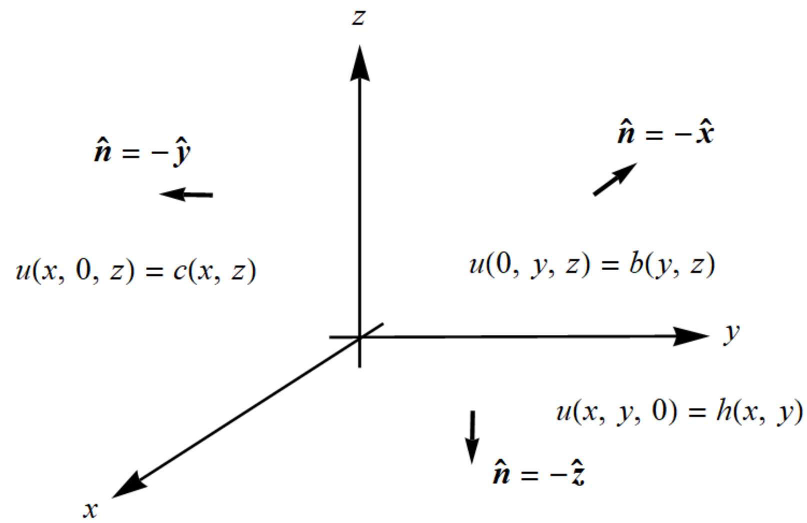
$$\begin{aligned} &\Delta G = \delta(x - x_0)\delta(y - y_0)\delta(z - z_0), \qquad x > 0,\ y > 0,\ z > 0 \\ &G = 0 \quad \text{on bdy } \mathscr{O}, \end{aligned}$$

where (x_0, y_0, z_0) is a point in the octant $\mathscr{O}$, then the identity becomes

$$\iiint_{\mathscr{O}} [u(x, y, z)\delta(x - x_0)\delta(y - y_0)\delta(z - z_0) - G(x, y, z; x_0, y_0, z_0)a(x, y, z)]\, dV = \iint_{\text{bdy } \mathscr{O}} \left(u\frac{\partial G}{\partial n} - (0)\frac{\partial u}{\partial n} \right) dS.$$

The normal derivative of G can be written as $\partial G/\partial n = \nabla G \cdot \hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is the outward unit vector normal to the boundary.

$$\iiint_{\mathscr{O}} u(x, y, z)\delta(x - x_0)\delta(y - y_0)\delta(z - z_0)\, dV - \iiint_{\mathscr{O}} G(x, y, z; x_0, y_0, z_0)a(x, y, z)\, dV = \iint_{\text{bdy } \mathscr{O}} u\nabla G \cdot \hat{\mathbf{n}}\, dS$$



Since (x_0, y_0, z_0) lies in the octant $\mathscr{O}$, the integral on the left involving the delta functions is $u(x_0, y_0, z_0)$. Also, integrating over the boundary results in three integrals, one over each of the yz-, xz-, and xy-faces.

$$\begin{aligned}
u(x_0, y_0, z_0) - \int_0^\infty \int_0^\infty \int_0^\infty G(x, y, z; x_0, y_0, z_0) a(x, y, z)\, dx\, dy\, dz \\
&= \int_0^\infty \int_0^\infty u(0, y, z) \nabla G \cdot (-\hat{\mathbf{x}}) \Big|_{x=0} dy\, dz + \int_0^\infty \int_0^\infty u(x, 0, z) \nabla G \cdot (-\hat{\mathbf{y}}) \Big|_{y=0} dx\, dz + \int_0^\infty \int_0^\infty u(x, y, 0) \nabla G \cdot (-\hat{\mathbf{z}}) \Big|_{z=0} dx\, dy \\
&= \int_0^\infty \int_0^\infty b(y, z) \left(-\frac{\partial G}{\partial x} \right) \Big|_{x=0} dy\, dz + \int_0^\infty \int_0^\infty c(x, z) \left(-\frac{\partial G}{\partial y} \right) \Big|_{y=0} dx\, dz + \int_0^\infty \int_0^\infty h(x, y) \left(-\frac{\partial G}{\partial z} \right) \Big|_{z=0} dx\, dy \\
&= -\int_0^\infty \int_0^\infty b(y, z) \frac{\partial G}{\partial x} \Big|_{x=0} dy\, dz - \int_0^\infty \int_0^\infty c(x, z) \frac{\partial G}{\partial y} \Big|_{y=0} dx\, dz - \int_0^\infty \int_0^\infty h(x, y) \frac{\partial G}{\partial z} \Big|_{z=0} dx\, dy
\end{aligned}$$

Solve this equation for u.

$$\begin{aligned}
u(x_0, y_0, z_0) = \int_0^\infty \int_0^\infty \int_0^\infty G(x, y, z; x_0, y_0, z_0) a(x, y, z)\, dx\, dy\, dz - \int_0^\infty \int_0^\infty b(y, z) \frac{\partial G}{\partial x} \Big|_{x=0} dy\, dz \\
- \int_0^\infty \int_0^\infty c(x, z) \frac{\partial G}{\partial y} \Big|_{y=0} dx\, dz - \int_0^\infty \int_0^\infty h(x, y) \frac{\partial G}{\partial z} \Big|_{z=0} dx\, dy
\end{aligned}$$

Switch the roles of x_0, y_0, and z_0 with those of x, y, and z, respectively.

$$u(x,y,z) = \int_0^\infty \int_0^\infty \int_0^\infty G(x_0,y_0,z_0;x,y,z)a(x_0,y_0,z_0)\,dx_0\,dy_0\,dz_0 - \int_0^\infty \int_0^\infty b(y_0,z_0)\frac{\partial G}{\partial x_0}\bigg|_{x_0=0} dy_0\,dz_0$$
$$- \int_0^\infty \int_0^\infty c(x_0,z_0)\frac{\partial G}{\partial y_0}\bigg|_{y_0=0} dx_0\,dz_0 - \int_0^\infty \int_0^\infty h(x_0,y_0)\frac{\partial G}{\partial z_0}\bigg|_{z_0=0} dx_0\,dy_0$$

Therefore, using the fact that the Green's function is symmetric,

$$\boxed{\begin{aligned} u(x,y,z) &= \int_0^\infty \int_0^\infty \int_0^\infty G(x,y,z;x_0,y_0,z_0)a(x_0,y_0,z_0)\,dx_0\,dy_0\,dz_0 \\ &\quad - \int_0^\infty \int_0^\infty b(y_0,z_0)\frac{\partial G}{\partial x_0}\bigg|_{x_0=0} dy_0\,dz_0 - \int_0^\infty \int_0^\infty c(x_0,z_0)\frac{\partial G}{\partial y_0}\bigg|_{y_0=0} dx_0\,dz_0 \\ &\quad - \int_0^\infty \int_0^\infty h(x_0,y_0)\frac{\partial G}{\partial z_0}\bigg|_{z_0=0} dx_0\,dy_0. \end{aligned}}$$

The solution for Poisson's equation is known, then, if the Green's function in the octant can be determined. Begin by finding the Green's function in infinite space (no boundaries).

$$\Delta g = \delta(x-x_0)\delta(y-y_0)\delta(z-z_0), \qquad -\infty < x,y,z < \infty$$

g can be interpreted as the electrostatic potential, and $\delta(x-x_0)\delta(y-y_0)\delta(z-z_0)$ can be interpreted as the charge density for a unit charge located at (x_0,y_0,z_0). Since there are no boundaries, g is expected to vary solely as a function of the radial distance from (x_0,y_0,z_0): $g = g(\mathscr{r})$, where $\mathscr{r} = \sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}$. Integrate both sides over a solid ball centered at (x_0,y_0,z_0) with radius $\mathscr{r}$.

$$\iiint_{\substack{(x-x_0)^2+(y-y_0)^2\\+(z-z_0)^2 \le \mathscr{r}^2}} \Delta g\,dV = \iiint_{\substack{(x-x_0)^2+(y-y_0)^2\\+(z-z_0)^2 \le \mathscr{r}^2}} \delta(x-x_0)\delta(y-y_0)\delta(z-z_0)\,dV$$

Since the ball contains (x_0,y_0,z_0), the right side is 1. Write the Laplacian operator Δ as ∇^2

$$\iiint_{\substack{(x-x_0)^2+(y-y_0)^2\\+(z-z_0)^2 \le \mathscr{r}^2}} \nabla^2 g\,dV = 1$$

$$\iiint_{\substack{(x-x_0)^2+(y-y_0)^2\\+(z-z_0)^2 \le \mathscr{r}^2}} \nabla \cdot \nabla g\,dV = 1$$

and apply the divergence theorem.

$$\iint_{\substack{(x-x_0)^2+(y-y_0)^2\\+(z-z_0)^2 = \mathscr{r}^2}} \nabla g \cdot \hat{\boldsymbol{\mathscr{r}}}\,dS = 1$$

Here $\hat{\boldsymbol{\mathscr{r}}}$ is the unit vector normal to this ball at every point on the boundary.

$$\iint_{\substack{(x-x_0)^2+(y-y_0)^2\\+(z-z_0)^2 = \mathscr{r}^2}} \frac{dg}{d\mathscr{r}}\,dS = 1$$

Because g only depends on $\mathscr{r}$, its derivative is constant on the ball's boundary.

$$\frac{dg}{d\mathscr{r}} \iint\limits_{\substack{(x-x_0)^2+(y-y_0)^2\\+(z-z_0)^2=\mathscr{r}^2}} dS = 1$$

This surface integral is just the ball's surface area.

$$\frac{dg}{d\mathscr{r}}(4\pi\mathscr{r}^2) = 1$$

Divide both sides by $4\pi\mathscr{r}^2$.

$$\frac{dg}{d\mathscr{r}} = \frac{1}{4\pi\mathscr{r}^2}$$

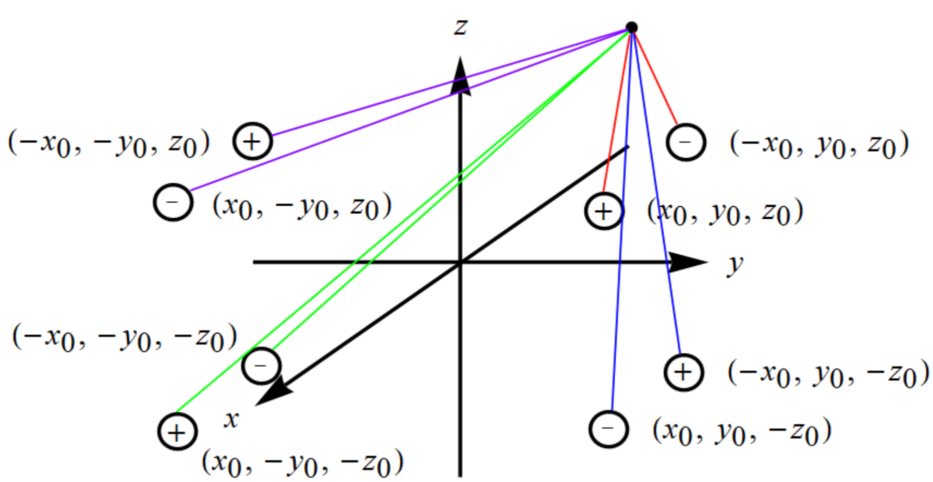
Integrate both sides with respect to $\mathscr{r}$.

$$g(\mathscr{r}) = -\frac{1}{4\pi\mathscr{r}}$$

The infinite-space Green's function is then

$$g(x,y,z;x_0,y_0,z_0) = -\frac{1}{4\pi\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}.$$

Now that it's known, the Green's function for the octant can be determined by the method of images. A convocation of point charges in space will be arranged so that the boundary condition, $G = 0$ on bdy $\mathscr{O}$, is satisfied. For a positive unit charge located at (x_0, y_0, z_0), alternating negative and positive unit charges should be placed in each octant at the reflection of (x_0, y_0, z_0). This way, the potential due to each positive charge is cancelled by that due to a negative charge at every point on the boundary.



The octant Green's function can now be written.

$$\begin{aligned} G(x,y,z;x_0,y_0,z_0) = &+g(x,y,z;x_0,y_0,z_0) - g(x,y,z;-x_0,y_0,z_0) - g(x,y,z;x_0,y_0,-z_0) \\ &+ g(x,y,z;-x_0,y_0,-z_0) - g(x,y,z;x_0,-y_0,z_0) + g(x,y,z;-x_0,-y_0,z_0) \\ &+ g(x,y,z;x_0,-y_0,-z_0) - g(x,y,z;-x_0,-y_0,-z_0), \quad x>0,\ y>0,\ z>0 \end{aligned}$$

Since g is defined over infinite space, it's important to note the restriction to $x > 0$, $y > 0$, $z > 0$ for G.

$$\begin{aligned}
G(x,y,z;x_0,y_0,z_0) = &-\frac{1}{4\pi\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}} + \frac{1}{4\pi\sqrt{(x+x_0)^2+(y-y_0)^2+(z-z_0)^2}} \\
&+ \frac{1}{4\pi\sqrt{(x-x_0)^2+(y-y_0)^2+(z+z_0)^2}} - \frac{1}{4\pi\sqrt{(x+x_0)^2+(y-y_0)^2+(z+z_0)^2}} \\
&+ \frac{1}{4\pi\sqrt{(x-x_0)^2+(y+y_0)^2+(z-z_0)^2}} - \frac{1}{4\pi\sqrt{(x+x_0)^2+(y+y_0)^2+(z-z_0)^2}} \\
&- \frac{1}{4\pi\sqrt{(x-x_0)^2+(y+y_0)^2+(z+z_0)^2}} + \frac{1}{4\pi\sqrt{(x+x_0)^2+(y+y_0)^2+(z+z_0)^2}}
\end{aligned}$$

Calculate $\partial G/\partial x_0$

$$\begin{aligned}
\frac{\partial G}{\partial x_0} = &-\frac{-2(x-x_0)\left(-\frac{1}{2}\right)}{4\pi[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]^{3/2}} + \frac{2(x+x_0)\left(-\frac{1}{2}\right)}{4\pi[(x+x_0)^2+(y-y_0)^2+(z-z_0)^2]^{3/2}} \\
&+ \frac{-2(x-x_0)\left(-\frac{1}{2}\right)}{4\pi[(x-x_0)^2+(y-y_0)^2+(z+z_0)^2]^{3/2}} - \frac{2(x+x_0)\left(-\frac{1}{2}\right)}{4\pi[(x+x_0)^2+(y-y_0)^2+(z+z_0)^2]^{3/2}} \\
&+ \frac{-2(x-x_0)\left(-\frac{1}{2}\right)}{4\pi[(x-x_0)^2+(y+y_0)^2+(z-z_0)^2]^{3/2}} - \frac{2(x+x_0)\left(-\frac{1}{2}\right)}{4\pi[(x+x_0)^2+(y+y_0)^2+(z-z_0)^2]^{3/2}} \\
&- \frac{-2(x-x_0)\left(-\frac{1}{2}\right)}{4\pi[(x-x_0)^2+(y+y_0)^2+(z+z_0)^2]^{3/2}} + \frac{2(x+x_0)\left(-\frac{1}{2}\right)}{4\pi[(x+x_0)^2+(y+y_0)^2+(z+z_0)^2]^{3/2}}
\end{aligned}$$

and evaluate it at $x_0 = 0$.

$$\begin{aligned}
\left.\frac{\partial G}{\partial x_0}\right|_{x_0=0} = &-\frac{x}{4\pi[x^2+(y-y_0)^2+(z-z_0)^2]^{3/2}} - \frac{x}{4\pi[x^2+(y-y_0)^2+(z-z_0)^2]^{3/2}} \\
&+ \frac{x}{4\pi[x^2+(y-y_0)^2+(z+z_0)^2]^{3/2}} + \frac{x}{4\pi[x^2+(y-y_0)^2+(z+z_0)^2]^{3/2}} \\
&+ \frac{x}{4\pi[x^2+(y+y_0)^2+(z-z_0)^2]^{3/2}} + \frac{x}{4\pi[x^2+(y+y_0)^2+(z-z_0)^2]^{3/2}} \\
&- \frac{x}{4\pi[x^2+(y+y_0)^2+(z+z_0)^2]^{3/2}} - \frac{x}{4\pi[x^2+(y+y_0)^2+(z+z_0)^2]^{3/2}}
\end{aligned}$$

Simplify the right side.

$$\left.\frac{\partial G}{\partial x_0}\right|_{x_0=0} = -\frac{x}{2\pi}\left\{\frac{1}{[x^2+(y-y_0)^2+(z-z_0)^2]^{3/2}} + \frac{1}{[x^2+(y+y_0)^2+(z+z_0)^2]^{3/2}} - \frac{1}{[x^2+(y-y_0)^2+(z+z_0)^2]^{3/2}} - \frac{1}{[x^2+(y+y_0)^2+(z-z_0)^2]^{3/2}}\right\}$$

Calculate $\partial G/\partial y_0$

$$\begin{aligned}\frac{\partial G}{\partial y_0} = &-\frac{-2(y-y_0)\left(-\frac{1}{2}\right)}{4\pi[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]^{3/2}} + \frac{-2(y-y_0)\left(-\frac{1}{2}\right)}{4\pi[(x+x_0)^2+(y-y_0)^2+(z-z_0)^2]^{3/2}} \\ &+ \frac{-2(y-y_0)\left(-\frac{1}{2}\right)}{4\pi[(x-x_0)^2+(y-y_0)^2+(z+z_0)^2]^{3/2}} - \frac{-2(y-y_0)\left(-\frac{1}{2}\right)}{4\pi[(x+x_0)^2+(y-y_0)^2+(z+z_0)^2]^{3/2}} \\ &+ \frac{2(y+y_0)\left(-\frac{1}{2}\right)}{4\pi[(x-x_0)^2+(y+y_0)^2+(z-z_0)^2]^{3/2}} - \frac{2(y+y_0)\left(-\frac{1}{2}\right)}{4\pi[(x+x_0)^2+(y+y_0)^2+(z-z_0)^2]^{3/2}} \\ &- \frac{2(y+y_0)\left(-\frac{1}{2}\right)}{4\pi[(x-x_0)^2+(y+y_0)^2+(z+z_0)^2]^{3/2}} + \frac{2(y+y_0)\left(-\frac{1}{2}\right)}{4\pi[(x+x_0)^2+(y+y_0)^2+(z+z_0)^2]^{3/2}}\end{aligned}$$

and evaluate it at $y_0 = 0$.

$$\begin{aligned}\left.\frac{\partial G}{\partial y_0}\right|_{y_0=0} &= -\frac{y}{4\pi[(x-x_0)^2+y^2+(z-z_0)^2]^{3/2}} + \frac{y}{4\pi[(x+x_0)^2+y^2+(z-z_0)^2]^{3/2}} \\ &\quad + \frac{y}{4\pi[(x-x_0)^2+y^2+(z+z_0)^2]^{3/2}} - \frac{y}{4\pi[(x+x_0)^2+y^2+(z+z_0)^2]^{3/2}} \\ &\quad - \frac{y}{4\pi[(x-x_0)^2+y^2+(z-z_0)^2]^{3/2}} + \frac{y}{4\pi[(x+x_0)^2+y^2+(z-z_0)^2]^{3/2}} \\ &\quad + \frac{y}{4\pi[(x-x_0)^2+y^2+(z+z_0)^2]^{3/2}} - \frac{y}{4\pi[(x+x_0)^2+y^2+(z+z_0)^2]^{3/2}} \\ &= -\frac{y}{2\pi}\left\{\frac{1}{[(x-x_0)^2+y^2+(z-z_0)^2]^{3/2}} + \frac{1}{[(x+x_0)^2+y^2+(z+z_0)^2]^{3/2}} - \frac{1}{[(x-x_0)^2+y^2+(z+z_0)^2]^{3/2}} - \frac{1}{[(x+x_0)^2+y^2+(z-z_0)^2]^{3/2}}\right\}\end{aligned}$$

Calculate $\partial G/\partial z_0$

$$\begin{aligned}\frac{\partial G}{\partial z_0} = &-\frac{-2(z-z_0)\left(-\frac{1}{2}\right)}{4\pi[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]^{3/2}} + \frac{-2(z-z_0)\left(-\frac{1}{2}\right)}{4\pi[(x+x_0)^2+(y-y_0)^2+(z-z_0)^2]^{3/2}} \\ &+ \frac{2(z+z_0)\left(-\frac{1}{2}\right)}{4\pi[(x-x_0)^2+(y-y_0)^2+(z+z_0)^2]^{3/2}} - \frac{2(z+z_0)\left(-\frac{1}{2}\right)}{4\pi[(x+x_0)^2+(y-y_0)^2+(z+z_0)^2]^{3/2}} \\ &+ \frac{-2(z-z_0)\left(-\frac{1}{2}\right)}{4\pi[(x-x_0)^2+(y+y_0)^2+(z-z_0)^2]^{3/2}} - \frac{-2(z-z_0)\left(-\frac{1}{2}\right)}{4\pi[(x+x_0)^2+(y+y_0)^2+(z-z_0)^2]^{3/2}} \\ &- \frac{2(z+z_0)\left(-\frac{1}{2}\right)}{4\pi[(x-x_0)^2+(y+y_0)^2+(z+z_0)^2]^{3/2}} + \frac{2(z+z_0)\left(-\frac{1}{2}\right)}{4\pi[(x+x_0)^2+(y+y_0)^2+(z+z_0)^2]^{3/2}}\end{aligned}$$

and evaluate it at $z_0 = 0$.

$$\begin{aligned}
\left.\frac{\partial G}{\partial z_0}\right|_{z_0=0} &= -\frac{z}{4\pi[(x-x_0)^2+(y-y_0)^2+z^2]^{3/2}} + \frac{z}{4\pi[(x+x_0)^2+(y-y_0)^2+z^2]^{3/2}} \\
&\quad - \frac{z}{4\pi[(x-x_0)^2+(y-y_0)^2+z^2]^{3/2}} + \frac{z}{4\pi[(x+x_0)^2+(y-y_0)^2+z^2]^{3/2}} \\
&\quad + \frac{z}{4\pi[(x-x_0)^2+(y+y_0)^2+z^2]^{3/2}} - \frac{z}{4\pi[(x+x_0)^2+(y+y_0)^2+z^2]^{3/2}} \\
&\quad + \frac{z}{4\pi[(x-x_0)^2+(y+y_0)^2+z^2]^{3/2}} - \frac{z}{4\pi[(x+x_0)^2+(y+y_0)^2+z^2]^{3/2}} \\
&= -\frac{z}{2\pi}\left\{\frac{1}{[(x-x_0)^2+(y-y_0)^2+z^2]^{3/2}} + \frac{1}{[(x+x_0)^2+(y+y_0)^2+z^2]^{3/2}}\right. \\
&\quad \left. - \frac{1}{[(x-x_0)^2+(y+y_0)^2+z^2]^{3/2}} - \frac{1}{[(x+x_0)^2+(y-y_0)^2+z^2]^{3/2}}\right\}
\end{aligned}$$

If $a = b = c = 0$, then the solution reduces to

$$\begin{aligned}
u(x,y,z) &= -\int_0^\infty\int_0^\infty h(x_0,y_0)\left.\frac{\partial G}{\partial z_0}\right|_{z_0=0} dx_0\,dy_0 \\
&= \frac{z}{2\pi}\int_0^\infty\int_0^\infty h(x_0,y_0)\left\{\frac{1}{[(x-x_0)^2+(y-y_0)^2+z^2]^{3/2}} + \frac{1}{[(x+x_0)^2+(y+y_0)^2+z^2]^{3/2}}\right. \\
&\quad \left. - \frac{1}{[(x-x_0)^2+(y+y_0)^2+z^2]^{3/2}} - \frac{1}{[(x+x_0)^2+(y-y_0)^2+z^2]^{3/2}}\right\} dx_0\,dy_0.
\end{aligned}$$