Exercise 18

(a) Find the Green’s function for the octant \( \mathcal{O} = \{(x, y, z) : x > 0, y > 0, z > 0\} \). (Hint: Use the method of reflection.)

(b) Use your answer in part (a) to solve the Dirichlet problem

\[
\begin{align*}
\begin{cases}
u_{xx} + u_{yy} + u_{zz} &= 0 \quad \text{in } \mathcal{O} \\
u(0, y, z) &= 0, \ u(x, 0, z) = 0, \ u(x, y, 0) = h(x, y) \quad \text{for } x > 0, \ y > 0, \ z > 0.
\end{cases}
\end{align*}
\]

Solution

The Poisson equation will be solved in the octant \( \mathcal{O} \) with three prescribed boundary conditions on the \( yz-, xz-, \) and \( xy\)-faces.

\[
\Delta u = a(x, y, z), \quad x > 0, \ y > 0, \ z > 0
\]

\[
u(0, y, z) = b(y, z)
\]

\[
u(x, 0, z) = c(x, z)
\]

\[
u(x, y, 0) = h(x, y)
\]

A Green’s function representation for the solution can be obtained from Green’s second identity,

\[
\iint\int_{\mathcal{O}} (u \Delta v - v \Delta u) \, dV = 
\iint_{\partial \mathcal{O}} \left( u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) \, dS.
\]

Let \( v = G = G(x, y, z; x_0, y_0, z_0) \) be the Green’s function.

\[
\iint\int_{\mathcal{O}} (u \Delta G - G \Delta u) \, dV = 
\iint_{\partial \mathcal{O}} \left( u \frac{\partial G}{\partial n} - G \frac{\partial u}{\partial n} \right) \, dS
\]

If we require it to satisfy

\[
\Delta G = \delta(x - x_0) \delta(y - y_0) \delta(z - z_0), \quad x > 0, \ y > 0, \ z > 0
\]

\[G = 0 \quad \text{on } \partial \mathcal{O},\]

where \((x_0, y_0, z_0)\) is a point in the octant \( \mathcal{O} \), then the identity becomes

\[
\iint\int_{\mathcal{O}} [u(x, y, z) \delta(x - x_0) \delta(y - y_0) \delta(z - z_0) - G(x, y, z; x_0, y_0, z_0) a(x, y, z)] \, dV = 
\iint_{\partial \mathcal{O}} \left( u \frac{\partial G}{\partial n} - 0 \frac{\partial u}{\partial n} \right) \, dS.
\]

The normal derivative of \( G \) can be written as \( \frac{\partial G}{\partial n} = \nabla G \cdot \hat{n} \), where \( \hat{n} \) is the outward unit vector normal to the boundary.

\[
\iint\int_{\mathcal{O}} u(x, y, z) \delta(x - x_0) \delta(y - y_0) \delta(z - z_0) \, dV - 
\iint\int_{\mathcal{O}} G(x, y, z; x_0, y_0, z_0) a(x, y, z) \, dV = 
\iint_{\partial \mathcal{O}} u \nabla G \cdot \hat{n} \, dS
\]

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Since \((x_0, y_0, z_0)\) lies in the octant \(O\), the integral on the left involving the delta functions is \(u(x_0, y_0, z_0)\). Also, integrating over the boundary results in three integrals, one over each of the \(yz\), \(xz\), and \(xy\)-faces.

\[
\begin{align*}
\hat{n} &= -\hat{y} \\
\hat{n} &= -\hat{x} \\
\hat{n} &= -\hat{z}
\end{align*}
\]

\[
\begin{align*}
u(x, 0, z) &= c(x, z) \\
u(0, y, z) &= b(y, z)
\end{align*}
\]

\[
\begin{align*}
u(x, y, 0) &= h(x, y)
\end{align*}
\]

\[
\begin{align*}
\hat{n} &= -\hat{y} \\
\hat{n} &= -\hat{x} \\
\hat{n} &= -\hat{z}
\end{align*}
\]

Solve this equation for \(u\).

\[
\begin{align*}
u(x_0, y_0, z_0) &= \int_0^\infty \int_0^\infty \int_0^\infty G(x, y, z; x_0, y_0, z_0) a(x, y, z) \, dx \, dy \, dz \\
&= \int_0^\infty \int_0^\infty \int_0^\infty u(0, y, z) \nabla G \cdot (-\hat{y}) \bigg|_{x=0} \, dy \, dz + \int_0^\infty \int_0^\infty \int_0^\infty u(x, 0, z) \nabla G \cdot (-\hat{y}) \bigg|_{y=0} \, dx \, dz + \int_0^\infty \int_0^\infty \int_0^\infty u(x, y, 0) \nabla G \cdot (-\hat{y}) \bigg|_{z=0} \, dx \, dy \\
&= \int_0^\infty \int_0^\infty b(y, z) \left( -\frac{\partial G}{\partial x} \right) \bigg|_{x=0} \, dy \, dz + \int_0^\infty \int_0^\infty c(x, z) \left( -\frac{\partial G}{\partial y} \right) \bigg|_{y=0} \, dx \, dz + \int_0^\infty \int_0^\infty h(x, y) \left( -\frac{\partial G}{\partial z} \right) \bigg|_{z=0} \, dx \, dy \\
&= -\int_0^\infty \int_0^\infty b(y, z) \frac{\partial G}{\partial x} \bigg|_{x=0} \, dy \, dz - \int_0^\infty \int_0^\infty c(x, z) \frac{\partial G}{\partial y} \bigg|_{y=0} \, dx \, dz - \int_0^\infty \int_0^\infty h(x, y) \frac{\partial G}{\partial z} \bigg|_{z=0} \, dx \, dy
\end{align*}
\]
Switch the roles of \( x_0, y_0, \) and \( z_0 \) with those of \( x, y, \) and \( z, \) respectively.

\[
u(x, y, z) = \int_0^\infty \int_0^\infty \int_0^\infty G(x_0, y_0, z_0; x, y, z) a(x_0, y_0, z_0) \, dx_0 \, dy_0 \, dz_0 - \int_0^\infty \int_0^\infty b(y_0, z_0) \frac{\partial G}{\partial x_0} \bigg|_{x_0=0} \, dy_0 \, dz_0
- \int_0^\infty \int_0^\infty c(x_0, z_0) \frac{\partial G}{\partial y_0} \bigg|_{y_0=0} \, dx_0 \, dz_0
- \int_0^\infty \int_0^\infty h(x_0, y_0) \frac{\partial G}{\partial z_0} \bigg|_{z_0=0} \, dx_0 \, dy_0.
\]

Therefore, using the fact that the Green’s function is symmetric,

\[
u(x, y, z) = \int_0^\infty \int_0^\infty \int_0^\infty G(x, y, z; x_0, y_0, z_0) a(x_0, y_0, z_0) \, dx_0 \, dy_0 \, dz_0
- \int_0^\infty \int_0^\infty b(y_0, z_0) \frac{\partial G}{\partial x_0} \bigg|_{x_0=0} \, dy_0 \, dz_0
- \int_0^\infty \int_0^\infty c(x_0, z_0) \frac{\partial G}{\partial y_0} \bigg|_{y_0=0} \, dx_0 \, dz_0
- \int_0^\infty \int_0^\infty h(x_0, y_0) \frac{\partial G}{\partial z_0} \bigg|_{z_0=0} \, dx_0 \, dy_0.
\]

The solution for Poisson’s equation is known, then, if the Green’s function in the octant can be determined. Begin by finding the Green’s function in infinite space (no boundaries).

\[\Delta g = \delta(x - x_0)\delta(y - y_0)\delta(z - z_0), \quad -\infty < x, y, z < \infty\]

g can be interpreted as the electrostatic potential, and \( \delta(x - x_0)\delta(y - y_0)\delta(z - z_0) \) can be interpreted as the charge density for a unit charge located at \( (x_0, y_0, z_0) \). Since there are no boundaries, \( g \) is expected to vary solely as a function of the radial distance from \( (x_0, y_0, z_0) \): \( g = g(\mathbf{r}) \), where \( \mathbf{r} = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} \). Integrate both sides over a solid ball centered at \( (x_0, y_0, z_0) \) with radius \( \mathbf{r} \).

\[\iiint_{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 \leq \mathbf{r}^2} \Delta g \, dV = \iiint_{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 \leq \mathbf{r}^2} \delta(x - x_0)\delta(y - y_0)\delta(z - z_0) \, dV\]

Since the ball contains \( (x_0, y_0, z_0) \), the right side is 1. Write the Laplacian operator \( \Delta \) as \( \nabla^2 \)

\[\iiint_{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 \leq \mathbf{r}^2} \nabla^2 g \, dV = 1\]

\[\iiint_{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 \leq \mathbf{r}^2} \nabla \cdot \nabla g \, dV = 1\]

and apply the divergence theorem.

\[\iiint_{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = \mathbf{r}^2} \nabla g \cdot \mathbf{\hat{n}} \, dS = 1\]

Here \( \mathbf{\hat{n}} \) is the unit vector normal to this ball at every point on the boundary.

\[\iiint_{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = \mathbf{r}^2} \frac{dg}{d\mathbf{r}} \, dS = 1\]

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Because $g$ only depends on $r$, its derivative is constant on the ball’s boundary.

$$
\frac{dg}{dr} \int \int \int \frac{dS}{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2} = 1
$$

This surface integral is just the ball’s surface area.

$$
\frac{dg}{dr} \left( \frac{4\pi r^2}{2} \right) = 1
$$

Divide both sides by $4\pi r^2$.

$$
\frac{dg}{dr} = \frac{1}{4\pi r^2}
$$

Integrate both sides with respect to $r$.

$$
g(r) = -\frac{1}{4\pi r}
$$

The infinite-space Green’s function is then

$$
g(x, y, z; x_0, y_0, z_0) = -\frac{1}{4\pi \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}}.
$$

Now that it’s known, the Green’s function for the octant can be determined by the method of images. A convocation of point charges in space will be arranged so that the boundary condition, $G = 0$ on bdy $\partial O$, is satisfied. For a positive unit charge located at $(x_0, y_0, z_0)$, alternating negative and positive unit charges should be placed in each octant at the reflection of $(x_0, y_0, z_0)$. This way, the potential due to each positive charge is cancelled by that due to a negative charge at every point on the boundary.

The octant Green’s function can now be written.

$$
G(x, y, z; x_0, y_0, z_0) = +g(x, y, z; x_0, y_0, z_0) - g(x, y, z; -x_0, y_0, z_0) - g(x, y, z; x_0, y_0, -z_0) \\
+ g(x, y, z; -x_0, y_0, -z_0) - g(x, y, z; x_0, -y_0, z_0) + g(x, y, z; -x_0, -y_0, z_0) \\
+ g(x, y, z; x_0, -y_0, -z_0) - g(x, y, z; -x_0, -y_0, -z_0), \quad x > 0, \ y > 0, \ z > 0
$$
Since \( g \) is defined over infinite space, it's important to note the restriction to \( x > 0, y > 0, z > 0 \) for \( G \).

\[
G(x, y, z; x_0, y_0, z_0) = -\frac{1}{4\pi\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}} + \frac{1}{4\pi\sqrt{(x+x_0)^2 + (y-y_0)^2 + (z-z_0)^2}}
\]

\[
+ \frac{1}{4\pi\sqrt{(x-x_0)^2 + (y+y_0)^2 + (z-z_0)^2}} - \frac{1}{4\pi\sqrt{(x+x_0)^2 + (y+y_0)^2 + (z-z_0)^2}}
\]

\[
- \frac{1}{4\pi\sqrt{(x-x_0)^2 + (y+y_0)^2 + (z+z_0)^2}} + \frac{1}{4\pi\sqrt{(x+x_0)^2 + (y+y_0)^2 + (z+z_0)^2}}
\]

Calculate \( \partial G / \partial x_0 \)

\[
\frac{\partial G}{\partial x_0} = -\frac{-2(x-x_0)\left(-\frac{1}{2}\right)}{4\pi[(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2]^{3/2}} + \frac{2(x+x_0)\left(-\frac{1}{2}\right)}{4\pi[(x+x_0)^2 + (y-y_0)^2 + (z-z_0)^2]^{3/2}}
\]

\[
+ \frac{-2(x-x_0)\left(-\frac{1}{2}\right)}{4\pi[(x-x_0)^2 + (y+y_0)^2 + (z-z_0)^2]^{3/2}} - \frac{2(x+x_0)\left(-\frac{1}{2}\right)}{4\pi[(x+x_0)^2 + (y+y_0)^2 + (z+z_0)^2]^{3/2}}
\]

\[
- \frac{-2(x-x_0)\left(-\frac{1}{2}\right)}{4\pi[(x-x_0)^2 + (y+y_0)^2 + (z+z_0)^2]^{3/2}} + \frac{2(x+x_0)\left(-\frac{1}{2}\right)}{4\pi[(x+x_0)^2 + (y+y_0)^2 + (z+z_0)^2]^{3/2}}
\]

and evaluate it at \( x_0 = 0 \).

\[
\left.\frac{\partial G}{\partial x_0}\right|_{x_0=0} = -\frac{x}{4\pi[x^2 + (y-y_0)^2 + (z-z_0)^2]^{3/2}} - \frac{x}{4\pi[x^2 + (y-y_0)^2 + (z-z_0)^2]^{3/2}}
\]

\[
+ \frac{x}{4\pi[x^2 + (y+y_0)^2 + (z-z_0)^2]^{3/2}} + \frac{x}{4\pi[x^2 + (y+y_0)^2 + (z+z_0)^2]^{3/2}}
\]

\[
- \frac{x}{4\pi[x^2 + (y+y_0)^2 + (z+z_0)^2]^{3/2}} + \frac{x}{4\pi[x^2 + (y+y_0)^2 + (z+z_0)^2]^{3/2}}
\]
Simplify the right side.

\[
\frac{\partial G}{\partial x_0} \bigg|_{x_0 = 0} = -\frac{x}{2\pi} \left\{ \frac{1}{[x^2 + (y-y_0)^2 + (z-z_0)^2]^{3/2}} + \frac{1}{[x^2 + (y+y_0)^2 + (z+z_0)^2]^{3/2}} - \frac{1}{[x^2 + (y-y_0)^2 + (z+z_0)^2]^{3/2}} - \frac{1}{[x^2 + (y+y_0)^2 + (z-z_0)^2]^{3/2}} \right\}
\]

Calculate \( \partial G / \partial y_0 \)

\[
\frac{\partial G}{\partial y_0} = -\frac{2(y-y_0) \left( -\frac{1}{2} \right)}{4\pi [(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2]^{3/2} + 4\pi [(x+x_0)^2 + (y-y_0)^2 + (z-z_0)^2]^{3/2}} + \frac{2(y+y_0) \left( -\frac{1}{2} \right)}{4\pi [(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2]^{3/2} - 4\pi [(x+x_0)^2 + (y+y_0)^2 + (z+z_0)^2]^{3/2}} - \frac{2(y-y_0) \left( -\frac{1}{2} \right)}{4\pi [(x-x_0)^2 + (y+y_0)^2 + (z-z_0)^2]^{3/2} + 4\pi [(x+x_0)^2 + (y+y_0)^2 + (z+z_0)^2]^{3/2}}
\]

and evaluate it at \( y_0 = 0 \).

\[
\frac{\partial G}{\partial y_0} \bigg|_{y_0 = 0} = -\frac{y}{4\pi [(x-x_0)^2 + y^2 + (z-z_0)^2]^{3/2} + 4\pi [(x+x_0)^2 + y^2 + (z-z_0)^2]^{3/2}} + \frac{y}{4\pi [(x-x_0)^2 + y^2 + (z-z_0)^2]^{3/2} - 4\pi [(x+x_0)^2 + y^2 + (z-z_0)^2]^{3/2}} + \frac{y}{4\pi [(x-x_0)^2 + y^2 + (z+z_0)^2]^{3/2} - 4\pi [(x+x_0)^2 + y^2 + (z+z_0)^2]^{3/2}}
\]

Calculate \( \partial G / \partial z_0 \)

\[
\frac{\partial G}{\partial z_0} = -\frac{2(z-z_0) \left( -\frac{1}{2} \right)}{4\pi [(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2]^{3/2} + 4\pi [(x+x_0)^2 + (y-y_0)^2 + (z-z_0)^2]^{3/2}} + \frac{2(z+z_0) \left( -\frac{1}{2} \right)}{4\pi [(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2]^{3/2} - 4\pi [(x+x_0)^2 + (y+y_0)^2 + (z-z_0)^2]^{3/2}} + \frac{2(z+z_0) \left( -\frac{1}{2} \right)}{4\pi [(x-x_0)^2 + (y+y_0)^2 + (z-z_0)^2]^{3/2} + 4\pi [(x+x_0)^2 + (y+y_0)^2 + (z+z_0)^2]^{3/2}}
\]
and evaluate it at $z_0 = 0$.

\[
\frac{\partial G}{\partial z_0} \bigg|_{z_0=0} = -\frac{z}{4\pi[(x-x_0)^2 + (y-y_0)^2 + z^2]^{3/2}} + \frac{z}{4\pi[(x+x_0)^2 + (y-y_0)^2 + z^2]^{3/2}}
\]

\[
- \frac{z}{4\pi[(x-x_0)^2 + (y-y_0)^2 + z^2]^{3/2}} + \frac{z}{4\pi[(x+x_0)^2 + (y-y_0)^2 + z^2]^{3/2}}
\]

\[
- \frac{z}{4\pi[(x-x_0)^2 + (y-y_0)^2 + z^2]^{3/2}} + \frac{z}{4\pi[(x+x_0)^2 + (y+y_0)^2 + z^2]^{3/2}}
\]

\[
- \frac{z}{4\pi[(x-x_0)^2 + (y-y_0)^2 + z^2]^{3/2}} + \frac{z}{4\pi[(x+x_0)^2 + (y+y_0)^2 + z^2]^{3/2}}
\]

\[
= -\frac{z}{2\pi} \left\{ \frac{1}{[(x-x_0)^2 + (y-y_0)^2 + z^2]^{3/2}} + \frac{1}{[(x+x_0)^2 + (y+y_0)^2 + z^2]^{3/2}}
\right\}
\]

If $a = b = c = 0$, then the solution reduces to

\[
u(x, y, z) = -\int_0^\infty \int_0^\infty h(x_0, y_0) \frac{\partial G}{\partial z_0} \bigg|_{z_0=0} \, dx_0 \, dy_0
\]

\[
= \frac{z}{2\pi} \int_0^\infty \int_0^\infty h(x_0, y_0) \left\{ \frac{1}{[(x-x_0)^2 + (y-y_0)^2 + z^2]^{3/2}} + \frac{1}{[(x+x_0)^2 + (y+y_0)^2 + z^2]^{3/2}}
\right\} \, dx_0 \, dy_0.
\]