Exercise 20

Use Exercise 19 to find the Green's function for the half-hyperspace $\{(x, y, z, w) : \ w > 0\}$.

Solution

Here the Poisson equation will be solved in the upper half-hyperspace D with a prescribed boundary condition on the xyz-hyperplane.

$$\begin{aligned} &\Delta u = f(x, y, z, w), \qquad -\infty < x, y, z < \infty, \ w > 0 \\ &u(x, y, z, 0) = F(x, y, z) \end{aligned}$$

A Green's function representation for the solution can be obtained from Green's second identity,

$$\iiiint_D (u\Delta v - v\Delta u)\, dV = \iiint_{\text{bdy } D} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) dS.$$

Let $v = G = G(x, y, z, w; x_0, y_0, z_0, w_0)$ be the Green's function.

$$\iiiint_D (u\Delta G - G\Delta u)\, dV = \iiint_{\text{bdy } D} \left(u\frac{\partial G}{\partial n} - G\frac{\partial u}{\partial n} \right) dS$$

If we require it to satisfy

$$\begin{aligned} &\Delta G = \delta(x - x_0)\delta(y - y_0)\delta(z - z_0)\delta(w - w_0), \qquad -\infty < x, y, z < \infty, \ w > 0 \\ &G = 0 \quad \text{on bdy } D, \end{aligned}$$

where (x_0, y_0, z_0, w_0) is a point in D, then the identity becomes

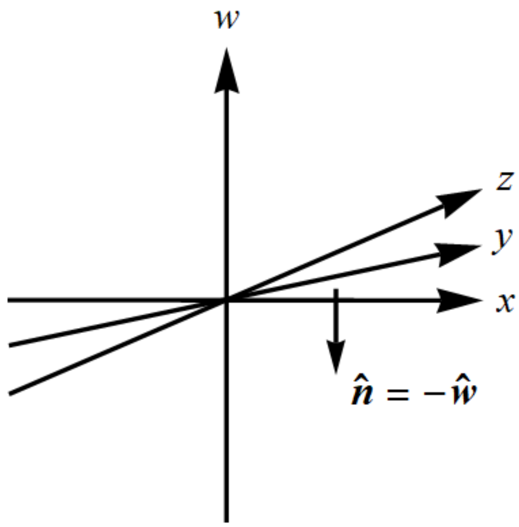
$$\iiiint_D [u(x, y, z, w)\delta(x-x_0)\delta(y-y_0)\delta(z-z_0)\delta(w-w_0) - G(x, y, z, w; x_0, y_0, z_0, w_0)f(x, y, z, w)]\, dV$$
$$= \iiint_{\text{bdy } D} \left(u\frac{\partial G}{\partial n} - (0)\frac{\partial u}{\partial n} \right) dS.$$

The normal derivative of G can be written as $\partial G/\partial n = \nabla G \cdot \hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is the outward unit vector normal to the boundary.

$$\iiiint_D u(x, y, z, w)\delta(x-x_0)\delta(y-y_0)\delta(z-z_0)\delta(w-w_0)\, dV - \iiiint_D G(x, y, z, w; x_0, y_0, z_0, w_0)f(x, y, z, w)\, dV$$
$$= \iiint_{\text{bdy } D} u\nabla G \cdot \hat{\mathbf{n}}\, dS$$

Since the point (x_0, y_0, z_0, w_0) is in D, the integral involving the delta functions is $u(x_0, y_0, z_0, w_0)$.

$$u(x_0, y_0, z_0, w_0) - \iiiint_D G(x, y, z, w; x_0, y_0, z_0, w_0)f(x, y, z, w)\, dV = \iiint_{\text{bdy } D} u\nabla G \cdot \hat{\mathbf{n}}\, dS$$



Simplify the right side.

$$\begin{aligned} u(x_0, y_0, z_0, w_0) - \iiiint_D G(x, y, z, w; x_0, y_0, z_0, w_0) f(x, y, z, w)\, dV &= \iiint_{\text{bdy } D} u \nabla G \cdot (-\hat{\mathbf{w}})\, dS \\ &= \iiint_{\text{bdy } D} u(x, y, z, 0) \left(-\frac{\partial G}{\partial w} \right) \bigg|_{w=0} dS \\ &= - \iiint_{\text{bdy } D} F(x, y, z) \frac{\partial G}{\partial w} \bigg|_{w=0} dS \end{aligned}$$

Solve for u.

$$\begin{aligned} u(x_0, y_0, z_0, w_0) = \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty G(x, y, z, w; x_0, y_0, z_0, w_0) f(x, y, z, w)\, dx\, dy\, dz\, dw \\ - \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty F(x, y, z) \frac{\partial G}{\partial w} \bigg|_{w=0} dx\, dy\, dz \end{aligned}$$

Switch the roles of x, y, z, and w with those of x_0, y_0, z_0, and w_0, respectively.

$$\begin{aligned} u(x, y, z, w) = \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty G(x_0, y_0, z_0, w_0; x, y, z, w) f(x_0, y_0, z_0, w_0)\, dx_0\, dy_0\, dz_0\, dw_0 \\ - \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty F(x_0, y_0, z_0) \frac{\partial G}{\partial w_0} \bigg|_{w_0=0} dx_0\, dy_0\, dz_0 \end{aligned}$$

Therefore, using the fact that the Green's function is symmetric,

$$\begin{aligned} u(x, y, z, w) = \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty G(x, y, z, w; x_0, y_0, z_0, w_0) f(x_0, y_0, z_0, w_0)\, dx_0\, dy_0\, dz_0\, dw_0 \\ - \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty F(x_0, y_0, z_0) \frac{\partial G}{\partial w_0} \bigg|_{w_0=0} dx_0\, dy_0\, dz_0. \end{aligned}$$

The solution for Poisson's equation is known, then, if the Green's function in the upper half-hyperspace can be determined.

Begin by finding the Green's function in infinite space (no boundaries).

$$\Delta g = \delta(x - x_0)\delta(y - y_0)\delta(z - z_0)\delta(w - w_0), \qquad -\infty < x, y, z, w < \infty$$

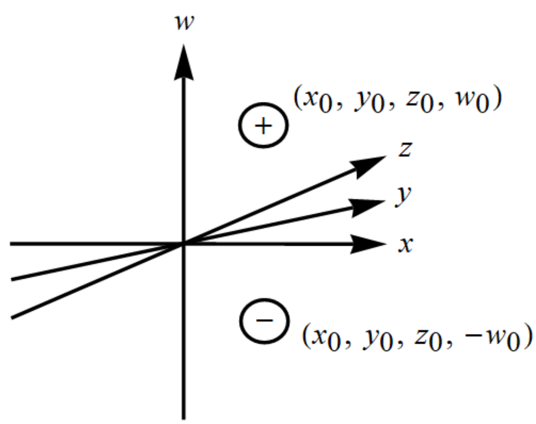
g can be interpreted as the electrostatic potential, and $\delta(x - x_0)\delta(y - y_0)\delta(z - z_0)\delta(w - w_0)$ can be interpreted as the charge density for a unit charge located at (x_0, y_0, z_0, w_0). Since there are no boundaries, g is expected to vary solely as a function of the radial distance from (x_0, y_0, z_0, w_0): $g = g(\imath)$, where $\imath = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 + (w - w_0)^2}$. In Exercise 19 it was found that the potential due to a unit charge at the origin in four-dimensional space is

$$g(x, y, z, w; 0, 0, 0, 0) = \frac{1}{x^2 + y^2 + z^2 + w^2}.$$

If this unit charge is at (x_0, y_0, z_0, w_0) instead, then

$$g(x, y, z, w; x_0, y_0, z_0, w_0) = \frac{1}{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 + (w - w_0)^2}$$

because of the translational invariance of the Laplacian. The method of images can now be applied to determine G, the Green's function in the upper half-hyperspace. A convocation of point charges in space will be arranged so that the boundary condition, $G = 0$ on bdy D, is satisfied.



If there's a positive unit charge at (x_0, y_0, z_0, w_0), place a negative unit charge at $(x_0, y_0, z_0, -w_0)$ so that every point on the xyz-hyperplane is equally spaced from them. The upper half-hyperspace Green's function can now be written.

$$G(x, y, z, w; x_0, y_0, z_0, w_0) = +g(x, y, z, w; x_0, y_0, z_0, w_0) - g(x, y, z, w; x_0, y_0, z_0, -w_0), \quad w > 0$$

Because g is defined over infinite four-dimensional space, it's important to note the restriction to $w > 0$ for G.

$$G(x, y, z, w; x_0, y_0, z_0, w_0) = \frac{1}{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 + (w - w_0)^2} - \frac{1}{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 + (w + w_0)^2}$$

Calculate $\partial G/\partial w_0$

$$\frac{\partial G}{\partial w_0} = -\frac{-2(w - w_0)}{[(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 + (w - w_0)^2]^2} + \frac{2(w + w_0)}{[(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 + (w + w_0)^2]^2}$$

and evaluate it at $w_0 = 0$.

$$\begin{aligned}\left.\frac{\partial G}{\partial w_0}\right|_{w_0=0} &= \frac{2w}{[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2+w^2]^2} + \frac{2w}{[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2+w^2]^2} \\ &= \frac{4w}{[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2+w^2]^2}\end{aligned}$$

Therefore,

$$\begin{aligned}u(x,y,z,w) = \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \Bigg[&\frac{f(x_0,y_0,z_0,w_0)}{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2+(w-w_0)^2} \\ &- \frac{f(x_0,y_0,z_0,w_0)}{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2+(w+w_0)^2}\Bigg]\,dx_0\,dy_0\,dz_0\,dw_0 \\ &- 4w\int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{F(x_0,y_0,z_0)}{[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2+w^2]^2}\,dx_0\,dy_0\,dz_0.\end{aligned}$$