Exercise 21

The Neumann function $N(x, y)$ for a domain $D$ is defined exactly like the Green’s function in Section 7.3 except that (ii) is replaced by the Neumann boundary condition

$$\frac{\partial N}{\partial n} = c \quad \text{for} \ x \in \text{bdy} \ D$$

for a suitable constant $c$.

(a) Show that $c = 1/A$, where $A$ is the area of bdy $D$. ($c = 0$ if $A = \infty$)

(b) State and prove the analog of Theorem 7.3.1, expressing the solution of the Neumann problem in terms of the Neumann function.

Solution

Consider the Poisson equation in some domain $D$ that is subject to a Neumann condition on the boundary of $D$.

$$\Delta U = f(x, y, z) \quad \text{in} \ D$$

$$\frac{\partial U}{\partial n} = g(x, y, z) \quad \text{on bdy} \ D$$

There is also a solvability condition for the boundary value problem, which is obtained by integrating both sides of the PDE over the volume of $D$.

$$\iiint_D \Delta U \, dV = \iiint_D f(x, y, z) \, dV$$

$$\iiint_D \nabla^2 U \, dV = \iiint_D f(x, y, z) \, dV$$

$$\iiint_D \nabla \cdot \nabla U \, dV = \iiint_D f(x, y, z) \, dV$$

Apply the divergence theorem on the left side. Let $\hat{n}$ be an outward unit vector normal to the boundary.

$$\iint_{\text{bdy} \, D} \nabla U \cdot \hat{n} \, dS = \iiint_D f(x, y, z) \, dV$$

$$\iint_{\text{bdy} \, D} \frac{\partial U}{\partial n} \, dS = \iiint_D f(x, y, z) \, dV$$

$$\iint_{\text{bdy} \, D} g(x, y, z) \, dS = \iiint_D f(x, y, z) \, dV$$

What this means is that the prescribed functions, $f$ and $g$, are not arbitrary and must satisfy this relationship for there to be a solution to the problem.

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Part (a)

Green’s second identity holds for any two functions, $u$ and $v$, on a domain $D$ and its boundary.

$$\iiint_D (u\Delta v - v\Delta u) \, dV = \iint_{\partial D} \left( u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) \, dS$$

Let $u = 1$ and let $v = N = N(x, y, z; x_0, y_0, z_0)$ be the Neumann function.

$$\iiint_D \Delta N \, dV = \iint_{\partial D} \frac{\partial N}{\partial n} \, dS$$

If we require the Neumann function to satisfy

$$\Delta N = \delta(x - x_0)\delta(y - y_0)\delta(z - z_0) \quad \text{in } D$$

$$\frac{\partial N}{\partial n} = c \quad \text{on } \partial D,$$

where $(x_0, y_0, z_0)$ is a point in $D$, then the identity reduces to

$$\iiint_D \delta(x - x_0)\delta(y - y_0)\delta(z - z_0) \, dV = \iint_{\partial D} c \, dS.$$

The volume integral of the delta functions is 1, and the constant $c$ can be brought in front of the surface integral.

$$1 = c \iint_{\partial D} dS$$

Therefore,

$$c = \frac{1}{\iint_{\partial D} dS} = \frac{1}{A},$$
Part (b)

A Neumann function representation for the solution can be obtained from Green's second identity,
\[
\iiint_D (u \Delta v - v \Delta u) \, dV = \iint_{\partial D} \left( u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) \, dS.
\]

Let \( u = U(x, y, z) \) and let \( v = N = N(x, y, z; x_0, y_0, z_0) \) be the Neumann function.
\[
\iiint_D (U \Delta N - N \Delta U) \, dV = \iint_{\partial D} \left( U \frac{\partial N}{\partial n} - N \frac{\partial U}{\partial n} \right) \, dS
\]

If we require \( N \) to satisfy
\[
\Delta N = \delta(x - x_0)\delta(y - y_0)\delta(z - z_0) \quad \text{in} \ D
\]
\[
\frac{\partial N}{\partial n} = c \quad \text{on} \ bdy \ D,
\]

where \((x_0, y_0, z_0)\) is a point in \(D\), then the identity becomes
\[
\iiint_D \left[ U(x, y, z)\delta(x - x_0)\delta(y - y_0)\delta(z - z_0) - N(x, y, z; x_0, y_0, z_0)f(x, y, z) \right] \, dV
\]
\[
= \iint_{\partial D} \left[ U(x, y, z)c - N(x, y, z; x_0, y_0, z_0)g(x, y, z) \right] \, dS.
\]

Split up the integrals.
\[
\iiint_D U(x, y, z)\delta(x - x_0)\delta(y - y_0)\delta(z - z_0) \, dV - \iiint_D N(x, y, z; x_0, y_0, z_0)f(x, y, z) \, dV
\]
\[
= c \iint_{\partial D} U(x, y, z) \, dS - \iint_{\partial D} N(x, y, z; x_0, y_0, z_0)g(x, y, z) \, dS
\]

The volume integral involving the delta functions is \(U(x_0, y_0, z_0)\). Rewrite \(c\) using the formula found for it in part (a).
\[
U(x_0, y_0, z_0) - \iiint_D N(x, y, z; x_0, y_0, z_0)f(x, y, z) \, dV
\]
\[
= \iint_{\partial D} U(x, y, z) \, dS - \iint_{\partial D} N(x, y, z; x_0, y_0, z_0)g(x, y, z) \, dS
\]

The first term on the right side is the average value of \(U\) over the boundary of \(D\), a constant. Denote it as \(\bar{U}\).
\[
U(x_0, y_0, z_0) - \iiint_D N(x, y, z; x_0, y_0, z_0)f(x, y, z) \, dV = \bar{U} - \iint_{\partial D} N(x, y, z; x_0, y_0, z_0)g(x, y, z) \, dS
\]

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Solve for $U$.

$$U(x_0, y_0, z_0) = \bar{U} + \iiint_D N(x, y, z; x_0, y_0, z_0) f(x, y, z) \, dV - \iint_{\partial D} N(x, y, z; x_0, y_0, z_0) g(x, y, z) \, dS$$

Switch the roles of $x_0$, $y_0$, and $z_0$ with those of $x$, $y$, and $z$, respectively.

$$U(x, y, z) = \bar{U} + \iiint_D N(x_0, y_0, z_0; x, y, z) f(x_0, y_0, z_0) \, dV_0 - \iint_{\partial D} N(x_0, y_0, z_0; x, y, z) g(x_0, y_0, z_0) \, dS_0$$