Exercise 22

Solve the Neumann problem in the half-plane: $\Delta u = 0$ in $\{y > 0\}$, $\partial u/\partial y = h(x)$ on $\{y = 0\}$ with $u(x, y)$ bounded at infinity. (*Hint:* Consider the problem satisfied by $v = \partial u/\partial y$.)

Solution

The Method of Images

The Poisson equation will be solved in the upper half-plane D with a Neumann boundary condition.

$$\begin{aligned} &\Delta U = f(x, y), \quad -\infty < x < \infty,\ y > 0 \\ &\frac{\partial U}{\partial y}(x, 0) = h(x) \end{aligned}$$

A Neumann function representation for the solution can be obtained from Green's second identity,

$$\iint_D (u\Delta v - v\Delta u)\, dA = \int_{\text{bdy } D} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) ds.$$

Let $u = U(x, y)$ and let $v = N = N(x, y; x_0, y_0)$ be the Neumann function.

$$\iint_D (U\Delta N - N\Delta U)\, dA = \int_{\text{bdy } D} \left(U\frac{\partial N}{\partial n} - N\frac{\partial U}{\partial n}\right) ds$$

If we require N to satisfy

$$\begin{aligned} &\Delta N = \delta(x - x_0)\delta(y - y_0), \quad -\infty < x < \infty,\ y > 0 \\ &\frac{\partial N}{\partial n} = c \quad \text{on bdy } D, \end{aligned}$$

where c is a constant and (x_0, y_0) is a point in the upper half-plane, then the identity becomes

$$\iint_D [U(x, y)\delta(x - x_0)\delta(y - y_0) - N(x, y; x_0, y_0)f(x, y)]\, dA = \int_{\text{bdy } D} \left[U(x, y)c - N(x, y; x_0, y_0)\frac{\partial U}{\partial n}\right] ds.$$

Write the normal derivative as $\partial U/\partial n = \nabla U \cdot \hat{\mathbf{n}}$ and split up the integrals.

$$\begin{aligned} &\iint_D U(x, y)\delta(x - x_0)\delta(y - y_0)\, dA - \iint_D N(x, y; x_0, y_0)f(x, y)\, dA \\ &\qquad = c\int_{\text{bdy } D} U(x, y)\, ds - \int_{\text{bdy } D} N(x, y; x_0, y_0)\nabla U \cdot \hat{\mathbf{n}}\, ds \end{aligned}$$

The integral involving the delta functions is $U(x_0, y_0)$.

$$U(x_0, y_0) - \iint_D N(x, y; x_0, y_0)f(x, y)\, dA = c\int_{\text{bdy } D} U(x, y)\, ds - \int_{\text{bdy } D} N(x, y; x_0, y_0)\nabla U \cdot \hat{\mathbf{n}}\, ds \quad (1)$$

Determine the constant c by setting $u = 1$ and $v = N(x, y; x_0, y_0)$ in Green's second identity.

$$\iint_D \Delta N \, dA = \int_{\text{bdy } D} \frac{\partial N}{\partial n} \, ds$$

$$\iint_D \delta(x - x_0)\delta(y - y_0) \, dA = \int_{\text{bdy } D} c \, ds$$

$$1 = c \int_{\text{bdy } D} ds$$

Solve for c.

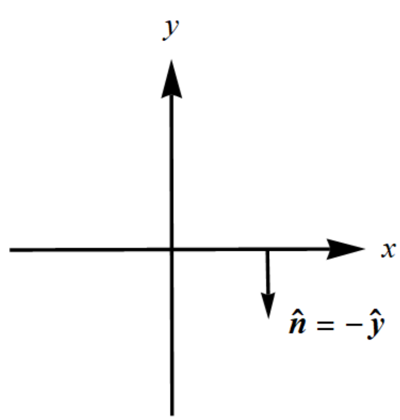
$$c = \frac{1}{\int_{\text{bdy } D} ds} = \frac{1}{\int_{-\infty}^{\infty} dx} = 0$$

Then equation (1) becomes

$$U(x_0, y_0) - \iint_D N(x, y; x_0, y_0) f(x, y) \, dA = \frac{\int_{\text{bdy } D} U(x, y) \, ds}{\int_{\text{bdy } D} ds} - \int_{\text{bdy } D} N(x, y; x_0, y_0) \nabla U \cdot \hat{\mathbf{n}} \, ds.$$

The first term on the right side is the average value of U on the boundary of the upper half-plane ($y = 0$), a constant. Denote it as $\overline{U}$.

$$U(x_0, y_0) - \iint_D N(x, y; x_0, y_0) f(x, y) \, dA = \overline{U} - \int_{\text{bdy } D} N(x, y; x_0, y_0) \nabla U \cdot \hat{\mathbf{n}} \, ds$$



As the figure illustrates, the outward unit vector normal to the upper half-plane is $\hat{\mathbf{n}} = -\hat{\mathbf{y}}$.

$$U(x_0, y_0) - \iint_D N(x, y; x_0, y_0) f(x, y) \, dA = \overline{U} - \int_{\text{bdy } D} N(x, y; x_0, y_0) \nabla U \cdot (-\hat{\mathbf{y}}) \, ds$$

Evaluate the dot product.

$$U(x_0, y_0) - \iint_D N(x, y; x_0, y_0) f(x, y)\, dA = \overline{U} - \int_{\text{bdy } D} N(x, y; x_0, y_0) \left(-\frac{\partial U}{\partial y}\right) ds$$

Substitute the prescribed boundary condition and write the integration limits.

$$U(x_0, y_0) - \int_0^\infty \int_{-\infty}^\infty N(x, y; x_0, y_0) f(x, y)\, dx\, dy = \overline{U} + \int_{-\infty}^\infty N(x, y; x_0, y_0)\big|_{y=0} h(x)\, dx$$

Solve for U.

$$U(x_0, y_0) = \overline{U} + \int_0^\infty \int_{-\infty}^\infty N(x, y; x_0, y_0) f(x, y)\, dx\, dy + \int_{-\infty}^\infty N(x, y; x_0, y_0)\big|_{y=0} h(x)\, dx$$

Switch the roles of x and y with those of x_0 and y_0, respectively.

$$U(x, y) = \overline{U} + \int_0^\infty \int_{-\infty}^\infty N(x_0, y_0; x, y) f(x_0, y_0)\, dx_0\, dy_0 + \int_{-\infty}^\infty N(x_0, y_0; x, y)\big|_{y_0=0} h(x_0)\, dx_0 \quad (2)$$

The Neumann function will now be shown to be symmetric if $c = 0$. Set $u = N(x, y; x_1, y_1)$ and $v = N(x, y; x_2, y_2)$ in Green's second identity,

$$\iint_D [N(x, y; x_1, y_1)\Delta N(x, y; x_2, y_2) - N(x, y; x_2, y_2)\Delta N(x, y; x_1, y_1)]\, dA$$
$$= \int_{\text{bdy } D} \left[N(x, y; x_1, y_1)\frac{\partial N}{\partial n}(x, y; x_2, y_2) - N(x, y; x_2, y_2)\frac{\partial N}{\partial n}(x, y; x_1, y_1)\right] ds,$$

where (x_1, y_1) and (x_2, y_2) are points in D, and $N(x, y; x_1, y_1)$ and $N(x, y; x_2, y_2)$ satisfy

$$\Delta N = \delta(x - x_1)\delta(y - y_1) \quad \text{in } D \qquad\qquad \Delta N = \delta(x - x_2)\delta(y - y_2) \quad \text{in } D$$
$$\frac{\partial N}{\partial n}(x, y; x_1, y_1) = c \quad \text{on bdy } D \qquad\qquad \frac{\partial N}{\partial n}(x, y; x_2, y_2) = c \quad \text{on bdy } D.$$

Substitute these results into the identity.

$$\iint_D [N(x, y; x_1, y_1)\delta(x - x_2)\delta(y - y_2) - N(x, y; x_2, y_2)\delta(x - x_1)\delta(y - y_1)]\, dA$$
$$= \int_{\text{bdy } D} [N(x, y; x_1, y_1)c - N(x, y; x_2, y_2)c]\, ds$$

Split up the integrals on the left and bring c in front of the integral on the right.

$$\iint_D N(x, y; x_1, y_1)\delta(x - x_2)\delta(y - y_2)\, dA - \iint_D N(x, y; x_2, y_2)\delta(x - x_1)\delta(y - y_1)\, dA$$
$$= c \int_{\text{bdy } D} [N(x, y; x_1, y_1) - N(x, y; x_2, y_2)]\, ds$$

If $c = 0$, then the right side is zero. Evaluate the integrals on the left.

$$N(x_2, y_2; x_1, y_1) - N(x_1, y_1; x_2, y_2) = 0$$

Therefore, $N(x_2, y_2; x_1, y_1) = N(x_1, y_1; x_2, y_2)$, and equation (2) becomes

$$U(x,y) = \overline{U} + \int_0^\infty \int_{-\infty}^\infty N(x, y; x_0, y_0) f(x_0, y_0)\, dx_0\, dy_0 + \int_{-\infty}^\infty N(x, y; x_0, y_0)\big|_{y_0=0}\, h(x_0) dx_0.$$

The solution for Poisson's equation is known, then, if the Neumann function in the upper half-plane can be determined. Begin by finding the Neumann function in the whole plane (no boundaries).

$$\Delta\mathscr{N} = \delta(x - x_0)\delta(y - y_0), \qquad -\infty < x < \infty,\ -\infty < y < \infty$$

$\mathscr{N}$ can be interpreted as the electrostatic potential, and $\delta(x-x_0)\delta(y-y_0)$ can be interpreted as the charge density for a unit charge located at (x_0, y_0). Since there are no boundaries, $\mathscr{N}$ is expected to vary solely as a function of the radial distance from (x_0, y_0): $\mathscr{N} = \mathscr{N}(\imath)$, where $\imath = \sqrt{(x-x_0)^2 + (y-y_0)^2}$. Integrate both sides over a disk centered at (x_0, y_0) with radius $\imath$.

$$\iint\limits_{(x-x_0)^2+(y-y_0)^2 \le \imath^2} \Delta\mathscr{N}\, dA = \iint\limits_{(x-x_0)^2+(y-y_0)^2 \le \imath^2} \delta(x-x_0)\delta(y-y_0)\, dA$$

Since the disk contains (x_0, y_0), the right side is 1. Write the Laplacian operator as $\Delta = \nabla^2$

$$\iint\limits_{(x-x_0)^2+(y-y_0)^2 \le \imath^2} \nabla^2\mathscr{N}\, dA = 1$$

$$\iint\limits_{(x-x_0)^2+(y-y_0)^2 \le \imath^2} \nabla\cdot\nabla\mathscr{N}\, dA = 1$$

and apply the two-dimensional divergence theorem.

$$\int\limits_{(x-x_0)^2+(y-y_0)^2 = \imath^2} \nabla\mathscr{N}\cdot\hat{\imath}\, ds = 1$$

Here $\hat{\imath}$ is the unit vector normal to this disk at every point on the boundary.

$$\int\limits_{(x-x_0)^2+(y-y_0)^2 = \imath^2} \frac{d\mathscr{N}}{d\imath}\, ds = 1$$

Because $\mathscr{N}$ only depends on $\imath$, its derivative is constant on the disk's boundary.

$$\frac{d\mathscr{N}}{d\imath} \int\limits_{(x-x_0)^2+(y-y_0)^2 = \imath^2} ds = 1$$

This line integral is just the disk's circumference.

$$\frac{d\mathscr{N}}{d\imath}(2\pi\imath) = 1$$

Divide both sides by $2\pi r$.

$$\frac{d\mathscr{N}}{dr} = \frac{1}{2\pi r}$$

Integrate both sides with respect to r.

$$\mathscr{N}(r) = \frac{1}{2\pi}\ln r$$

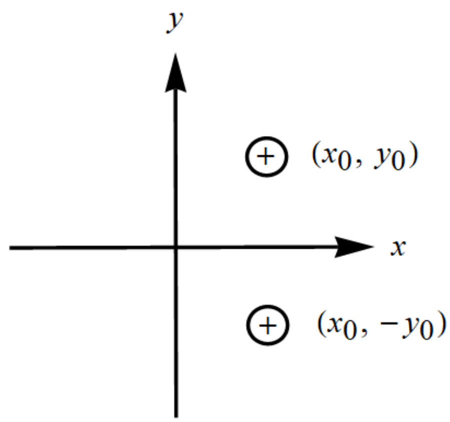
The Neumann function for the whole plane is then

$$\mathscr{N}(x, y; x_0, y_0) = \frac{1}{2\pi}\ln\sqrt{(x-x_0)^2 + (y-y_0)^2}.$$

Now that it's known, the Neumann function for the upper half-plane can be determined by the method of images. A convocation of point charges in the whole plane will be arranged so that the boundary condition,

$$\frac{\partial N}{\partial n} = c = \frac{1}{\displaystyle\int_{\text{bdy } D} ds} = \frac{1}{\displaystyle\int_{-\infty}^{\infty} dx} = 0 \text{ on bdy } D \quad \Rightarrow \quad \frac{\partial N}{\partial y}(x, 0) = 0,$$

is satisfied. This derivative of potential can be interpreted as the y-component of the electric field. For a positive unit charge at (x_0, y_0), place another positive unit charge at $(x_0, -y_0)$ so that every point on the x-axis is equally spaced from both.



The upper half-plane Neumann function can now be written.

$$N(x, y; x_0, y_0) = +\mathscr{N}(x, y; x_0, y_0) + \mathscr{N}(x, y; x_0, -y_0), \quad y > 0$$

Because $\mathscr{N}$ is defined over the whole plane, it's important to note the restriction to $y > 0$ for N.

$$\begin{aligned} N(x, y; x_0, y_0) &= \frac{1}{2\pi}\ln\sqrt{(x-x_0)^2 + (y-y_0)^2} + \frac{1}{2\pi}\ln\sqrt{(x-x_0)^2 + (y+y_0)^2} \\ &= \frac{1}{2\pi}\ln\left[\sqrt{(x-x_0)^2 + (y-y_0)^2}\sqrt{(x-x_0)^2 + (y+y_0)^2}\right] \\ &= \frac{1}{4\pi}\ln\left\{[(x-x_0)^2 + (y-y_0)^2][(x-x_0)^2 + (y+y_0)^2]\right\} \end{aligned}$$

Now set $y_0 = 0$ in this result.

$$\begin{aligned} N(x,y;x_0,y_0)\big|_{y_0=0} &= \frac{1}{4\pi}\ln\left\{[(x-x_0)^2+y^2][(x-x_0)^2+y^2]\right\} \\ &= \frac{1}{4\pi}\ln[(x-x_0)^2+y^2]^2 \\ &= \frac{1}{2\pi}\ln[(x-x_0)^2+y^2] \end{aligned}$$

Therefore, the solution to Poisson's equation in the upper half-plane with a Neumann boundary condition is

$$\begin{aligned} U(x,y) = \overline{U} + \frac{1}{4\pi}\int_0^\infty \int_{-\infty}^\infty f(x_0,y_0)\ln\left\{[(x-x_0)^2+(y-y_0)^2][(x-x_0)^2+(y+y_0)^2]\right\}dx_0\,dy_0 \\ + \frac{1}{2\pi}\int_{-\infty}^\infty h(x_0)\ln[(x-x_0)^2+y^2]\,dx_0. \end{aligned}$$

If $f = 0$, then the solution reduces to

$$U(x,y) = \overline{U} + \frac{1}{2\pi}\int_{-\infty}^\infty h(x_0)\ln[(x-x_0)^2+y^2]\,dx_0.$$

This answer is in disagreement with the one at the back of the book.

Using Exercise 6

This answer will be verified by solving the Laplace equation in the upper half-plane with a Neumann boundary condition in a different way.

$$\begin{aligned} &\Delta U = 0, \quad -\infty < x < \infty,\ y > 0 \\ &\frac{\partial U}{\partial y}(x,0) = h(x) \end{aligned}$$

Differentiate both sides of the PDE with respect to y.

$$\frac{\partial}{\partial y}(\Delta U) = \frac{\partial}{\partial y}(0)$$

$$\frac{\partial}{\partial y}\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right) = 0$$

$$\frac{\partial}{\partial y}\left(\frac{\partial^2 U}{\partial x^2}\right) + \frac{\partial}{\partial y}\left(\frac{\partial^2 U}{\partial y^2}\right) = 0$$

The mixed derivatives are equal by Clairaut's theorem.

$$\frac{\partial^2}{\partial x^2}\left(\frac{\partial U}{\partial y}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial U}{\partial y}\right) = 0$$

Make the substitution $V(x,y) = \partial U/\partial y$ in the PDE and its associated boundary condition.

$$\begin{aligned} &\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \\ &V(x,0) = h(x) \end{aligned}$$

The problem becomes the same one that was solved in Exercise 6.

$$V(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{h(x_0)}{(x_0-x)^2+y^2}\,dx_0$$

Now that the solution is known, change back to U.

$$\frac{\partial U}{\partial y} = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{h(x_0)}{(x_0-x)^2+y^2}\,dx_0$$

Integrate both sides partially with respect to y.

$$\begin{aligned}
U(x,y) &= \int^{y}\left[\frac{s}{\pi}\int_{-\infty}^{\infty}\frac{h(x_0)}{(x_0-x)^2+s^2}\,dx_0\right]ds + F(x) \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty}\left[\int^{y}\frac{s}{(x_0-x)^2+s^2}\,ds\right]h(x_0)\,dx_0 + F(x)
\end{aligned}$$

Let

$$\begin{aligned}
w &= (x_0-x)^2+s^2 \\
dw &= 2s\,ds \quad \rightarrow \quad \frac{dw}{2} = s\,ds.
\end{aligned}$$

Consequently,

$$\begin{aligned}
U(x,y) &= \frac{1}{\pi}\int_{-\infty}^{\infty}\left[\int^{(x_0-x)^2+y^2}\frac{1}{w}\frac{dw}{2}\right]h(x_0)\,dx_0 + F(x) \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty}\left\{\frac{1}{2}\ln[(x_0-x)^2+y^2]\right\}h(x_0)\,dx_0 + F(x) \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}h(x_0)\ln[(x_0-x)^2+y^2]\,dx_0 + F(x).
\end{aligned}$$