Exercise 22

Solve the Neumann problem in the half-plane: \( \Delta u = 0 \) in \( \{ y > 0 \} \), \( \partial u / \partial y = h(x) \) on \( \{ y = 0 \} \) with \( u(x, y) \) bounded at infinity. (Hint: Consider the problem satisfied by \( v = \partial u / \partial y \).)

Solution

The Method of Images

The Poisson equation will be solved in the upper half-plane \( D \) with a Neumann boundary condition.

\[
\Delta U = f(x, y), \quad -\infty < x < \infty, \quad y > 0
\]

\[
\frac{\partial U}{\partial y}(x, 0) = h(x)
\]

A Neumann function representation for the solution can be obtained from Green’s second identity,

\[
\iint_D (u \Delta v - v \Delta u) \, dA = \int_{\text{bdy } D} \left( \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) \, ds.
\]

Let \( u = U(x, y) \) and let \( v = N = N(x, y; x_0, y_0) \) be the Neumann function.

\[
\iint_D (U \Delta N - N \Delta U) \, dA = \int_{\text{bdy } D} \left( U \frac{\partial N}{\partial n} - N \frac{\partial U}{\partial n} \right) \, ds
\]

If we require \( N \) to satisfy

\[
\Delta N = \delta(x - x_0) \delta(y - y_0), \quad -\infty < x < \infty, \quad y > 0
\]

\[
\frac{\partial N}{\partial n} = c \quad \text{on bdy } D,
\]

where \( c \) is a constant and \( (x_0, y_0) \) is a point in the upper half-plane, then the identity becomes

\[
\iint_D [U(x, y) \delta(x - x_0) \delta(y - y_0) - N(x, y; x_0, y_0) f(x, y)] \, dA = \int_{\text{bdy } D} \left[ U(x, y) c - N(x, y; x_0, y_0) \frac{\partial U}{\partial n} \right] \, ds.
\]

Write the normal derivative as \( \partial U / \partial n = \nabla U \cdot \hat{n} \) and split up the integrals.

\[
\iint_D U(x, y) \delta(x - x_0) \delta(y - y_0) \, dA - \iint_D N(x, y; x_0, y_0) f(x, y) \, dA
\]

\[= c \int_{\text{bdy } D} U(x, y) \, ds - \int_{\text{bdy } D} N(x, y; x_0, y_0) \nabla U \cdot \hat{n} \, ds
\]

The integral involving the delta functions is \( U(x_0, y_0) \).

\[
U(x_0, y_0) - \iint_D N(x, y; x_0, y_0) f(x, y) \, dA = c \int_{\text{bdy } D} U(x, y) \, ds - \int_{\text{bdy } D} N(x, y; x_0, y_0) \nabla U \cdot \hat{n} \, ds \quad (1)
\]

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Determine the constant $c$ by setting $u = 1$ and $v = N(x, y; x_0, y_0)$ in Green’s second identity.

\[
\oint_D \Delta N \, dA = \int_{\text{bdy } D} \frac{\partial N}{\partial n} \, ds
\]

\[
\oint_D \delta(x - x_0)\delta(y - y_0) \, dA = \int_{\text{bdy } D} c \, ds
\]

\[
1 = c \int_{\text{bdy } D} ds
\]

Solve for $c$.

\[
c = \frac{1}{\int_{\text{bdy } D} ds} = \frac{1}{\int_{-\infty}^{\infty} dx} = 0
\]

Then equation (1) becomes

\[
U(x_0, y_0) - \oint_D N(x, y; x_0, y_0)f(x, y) \, dA = \int_{\text{bdy } D} U(x, y) \, ds
\]

\[
\oint_{\text{bdy } D} N(x, y; x_0, y_0)\nabla U \cdot \hat{n} \, ds.
\]

The first term on the right side is the average value of $U$ on the boundary of the upper half-plane ($y = 0$), a constant. Denote it as $\bar{U}$.

\[
U(x_0, y_0) - \oint_D N(x, y; x_0, y_0)f(x, y) \, dA = \bar{U} - \int_{\text{bdy } D} N(x, y; x_0, y_0)\nabla U \cdot \hat{n} \, ds
\]

As the figure illustrates, the outward unit vector normal to the upper half-plane is $\hat{n} = -\hat{y}$.

\[
U(x_0, y_0) - \oint_D N(x, y; x_0, y_0)f(x, y) \, dA = \bar{U} - \int_{\text{bdy } D} N(x, y; x_0, y_0)\nabla U \cdot (-\hat{y}) \, ds
\]
Evaluate the dot product.

\[ U(x_0, y_0) - \iint_D N(x, y; x_0, y_0)f(x, y)\,dA = \overline{U} - \int_{\overline{\partial D}} N(x, y; x_0, y_0) \left( -\frac{\partial U}{\partial y} \right) ds \]

Substitute the prescribed boundary condition and write the integration limits.

\[ U(x_0, y_0) - \int_0^\infty \int_{-\infty}^\infty N(x, y; x_0, y_0)f(x, y)\,dx\,dy = \overline{U} + \int_{-\infty}^\infty N(x, y; x_0, y_0)|_{y=0} h(x)\,dx \]

Solve for \( U \).

\[ U(x_0, y_0) = \overline{U} + \int_0^\infty \int_{-\infty}^\infty N(x, y; x_0, y_0)f(x, y)\,dx\,dy + \int_{-\infty}^\infty N(x_0, y_0; x, y)|_{y=0} h(x)\,dx \]

Switch the roles of \( x \) and \( y \) with those of \( x_0 \) and \( y_0 \), respectively.

\[ U(x, y) = \overline{U} + \int_0^\infty \int_{-\infty}^\infty N(x_0, y_0; x, y)f(x_0, y_0)\,dx_0\,dy_0 + \int_{-\infty}^\infty N(x_0, y_0; x, y)|_{y_0=0} h(x_0)\,dx_0 \quad (2) \]

The Neumann function will now be shown to be symmetric if \( c = 0 \). Set \( u = N(x, y; x_1, y_1) \) and \( v = N(x, y; x_2, y_2) \) in Green’s second identity,

\[ \iint_D [N(x, y; x_1, y_1)\Delta N(x, y; x_2, y_2) - N(x, y; x_2, y_2)\Delta N(x, y; x_1, y_1)]\,dA = \int_{\overline{\partial D}} \left[ N(x, y; x_1, y_1) \frac{\partial N}{\partial n}(x, y; x_2, y_2) - N(x, y; x_2, y_2) \frac{\partial N}{\partial n}(x, y; x_1, y_1) \right] ds, \]

where \((x_1, y_1)\) and \((x_2, y_2)\) are points in \( D \), and \( N(x, y; x_1, y_1) \) and \( N(x, y; x_2, y_2) \) satisfy

\[
\Delta N = \delta(x - x_1)\delta(y - y_1) \quad \text{in } D \quad \Delta N = \delta(x - x_2)\delta(y - y_2) \quad \text{in } D \\
\frac{\partial N}{\partial n}(x, y; x_1, y_1) = c \quad \text{on } \overline{\partial D} \quad \frac{\partial N}{\partial n}(x, y; x_2, y_2) = c \quad \text{on } \overline{\partial D}.
\]

Substitute these results into the identity.

\[ \iint_D [N(x, y; x_1, y_1)\delta(x - x_2)\delta(y - y_2) - N(x, y; x_2, y_2)\delta(x - x_1)\delta(y - y_1)]\,dA = \int_{\overline{\partial D}} [N(x, y; x_1, y_1)c - N(x, y; x_2, y_2)c]\,ds \]

Split up the integrals on the left and bring \( c \) in front of the integral on the right.

\[ \iint_D N(x, y; x_1, y_1)\delta(x - x_2)\delta(y - y_2)\,dA - \iint_D N(x, y; x_2, y_2)\delta(x - x_1)\delta(y - y_1)\,dA = c \int_{\overline{\partial D}} [N(x, y; x_1, y_1) - N(x, y; x_2, y_2)]\,ds \]
If \( c = 0 \), then the right side is zero. Evaluate the integrals on the left.

\[ N(x_2, y_2; x_1, y_1) - N(x_1, y_1; x_2, y_2) = 0 \]

Therefore, \( N(x_2, y_2; x_1, y_1) = N(x_1, y_1; x_2, y_2) \), and equation (2) becomes

\[ U(x, y) = \overline{U} + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} N(x, y; x_0, y_0) f(x_0, y_0) \, dx_0 \, dy_0 + \int_{-\infty}^{\infty} N(x, y; x_0, y_0) |_{y_0=0} \, h(x_0) \, dx_0. \]

The solution for Poisson’s equation is known, then, if the Neumann function in the upper half-plane can be determined. Begin by finding the Neumann function in the whole plane (no boundaries).

\[ \Delta \mathcal{N} = \delta(x - x_0) \delta(y - y_0), \quad -\infty < x < \infty, \quad -\infty < y < \infty \]

\( \mathcal{N} \) can be interpreted as the electrostatic potential, and \( \delta(x - x_0) \delta(y - y_0) \) can be interpreted as the charge density for a unit charge located at \((x_0, y_0)\). Since there are no boundaries, \( \mathcal{N} \) is expected to vary solely as a function of the radial distance from \((x_0, y_0)\):

\[ \mathcal{N} = \mathcal{N}(r), \quad \text{where} \quad r = \sqrt{(x - x_0)^2 + (y - y_0)^2}. \]

Integrate both sides over a disk centered at \((x_0, y_0)\) with radius \( r \).

\[ \frac{\partial}{\partial r} \int_{(x-x_0)^2+(y-y_0)^2 \leq r^2} \Delta \mathcal{N} \, dA = \frac{\partial}{\partial r} \int_{(x-x_0)^2+(y-y_0)^2 \leq r^2} \delta(x - x_0) \delta(y - y_0) \, dA \]

Since the disk contains \((x_0, y_0)\), the right side is 1. Write the Laplacian operator as \( \Delta = \nabla^2 \)

\[ \frac{\partial}{\partial r} \int_{(x-x_0)^2+(y-y_0)^2 \leq r^2} \nabla^2 \mathcal{N} \, dA = 1 \]

\[ \frac{\partial}{\partial r} \int_{(x-x_0)^2+(y-y_0)^2 \leq r^2} \nabla \cdot \nabla \mathcal{N} \, dA = 1 \]

and apply the two-dimensional divergence theorem.

\[ \int_{(x-x_0)^2+(y-y_0)^2 = r^2} \nabla \mathcal{N} \cdot \hat{n} \, ds = 1 \]

Here \( \hat{n} \) is the unit vector normal to this disk at every point on the boundary.

\[ \int_{(x-x_0)^2+(y-y_0)^2 = r^2} \frac{d\mathcal{N}}{d\varepsilon} \, ds = 1 \]

Because \( \mathcal{N} \) only depends on \( r \), its derivative is constant on the disk’s boundary.

\[ \frac{d\mathcal{N}}{d\varepsilon} \int_{(x-x_0)^2+(y-y_0)^2 = r^2} ds = 1 \]

This line integral is just the disk’s circumference.

\[ \frac{d\mathcal{N}}{d\varepsilon} (2\pi r) = 1 \]
Divide both sides by $2\pi \varepsilon$.

\[ \frac{dN}{d\varepsilon} = \frac{1}{2\pi \varepsilon} \]

Integrate both sides with respect to $\varepsilon$.

\[ N(\varepsilon) = \frac{1}{2\pi} \ln \varepsilon \]

The Neumann function for the whole plane is then

\[ N(x, y; x_0, y_0) = \frac{1}{2\pi} \ln \sqrt{(x-x_0)^2 + (y-y_0)^2}. \]

Now that it’s known, the Neumann function for the upper half-plane can be determined by the method of images. A convocation of point charges in the whole plane will be arranged so that the boundary condition,

\[ \frac{\partial N}{\partial n} = c = \frac{1}{\int_{\text{bdy } D} ds} = \frac{1}{\int_{-\infty}^{\infty} dx} = 0 \text{ on bdy } D \Rightarrow \frac{\partial N}{\partial y}(x, 0) = 0, \]

is satisfied. This derivative of potential can be interpreted as the $y$-component of the electric field. For a positive unit charge at $(x_0, y_0)$, place another positive unit charge at $(x_0, -y_0)$ so that every point on the $x$-axis is equally spaced from both.

The upper half-plane Neumann function can now be written.

\[ N(x, y; x_0, y_0) = +N(x, y; x_0, y_0) + N(x, y; x_0, -y_0), \quad y > 0 \]

Because $N$ is defined over the whole plane, it’s important to note the restriction to $y > 0$ for $N$.

\[
N(x, y; x_0, y_0) = \frac{1}{2\pi} \ln \sqrt{(x-x_0)^2 + (y-y_0)^2} + \frac{1}{2\pi} \ln \sqrt{(x-x_0)^2 + (y+y_0)^2} \\
= \frac{1}{2\pi} \ln \left[ \sqrt{(x-x_0)^2 + (y-y_0)^2} \sqrt{(x-x_0)^2 + (y+y_0)^2} \right] \\
= \frac{1}{4\pi} \ln \left\{ \frac{(x-x_0)^2 + (y-y_0)^2}{(x-x_0)^2 + (y+y_0)^2} \right\}
\]
Now set \( y_0 = 0 \) in this result.

\[
N(x, y; x_0, y_0)|_{y_0=0} = \frac{1}{4\pi} \ln \left\{ [(x - x_0)^2 + y^2][x - x_0]^2 + y^2] \right\}
= \frac{1}{4\pi} \ln[(x - x_0)^2 + y^2]^2
= \frac{1}{2\pi} \ln[(x - x_0)^2 + y^2]
\]

Therefore, the solution to Poisson’s equation in the upper half-plane with a Neumann boundary condition is

\[
U(x, y) = \bar{U} + \frac{1}{4\pi} \int_0^\infty \int_{-\infty}^\infty f(x_0, y_0) \ln \left\{ [(x - x_0)^2 + (y - y_0)^2][(x - x_0)^2 + (y + y_0)^2] \right\} \, dx_0 \, dy_0
+ \frac{1}{2\pi} \int_{-\infty}^\infty h(x_0) \ln[(x - x_0)^2 + y^2] \, dx_0.
\]

If \( f = 0 \), then the solution reduces to

\[
U(x, y) = \bar{U} + \frac{1}{2\pi} \int_{-\infty}^\infty h(x_0) \ln[(x - x_0)^2 + y^2] \, dx_0.
\]

This answer is in disagreement with the one at the back of the book.

**Using Exercise 6**

This answer will be verified by solving the Laplace equation in the upper half-plane with a Neumann boundary condition in a different way.

\[
\Delta U = 0, \quad -\infty < x < \infty, \quad y > 0
\]

\[
\frac{\partial U}{\partial y}(x, 0) = h(x)
\]

Differentiate both sides of the PDE with respect to \( y \).

\[
\frac{\partial}{\partial y}(\Delta U) = \frac{\partial}{\partial y}(0)
\]

\[
\frac{\partial}{\partial y} \left( \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right) = 0
\]

\[
\frac{\partial}{\partial y} \left( \frac{\partial^2 U}{\partial x^2} \right) + \frac{\partial}{\partial y} \left( \frac{\partial^2 U}{\partial y^2} \right) = 0
\]

The mixed derivatives are equal by Clairaut’s theorem.

\[
\frac{\partial^2}{\partial x^2} \left( \frac{\partial U}{\partial y} \right) + \frac{\partial^2}{\partial y^2} \left( \frac{\partial U}{\partial y} \right) = 0
\]

Make the substitution \( V(x, y) = \partial U/\partial y \) in the PDE and its associated boundary condition.

\[
\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0
\]

\[
V(x, 0) = h(x)
\]
The problem becomes the same one that was solved in Exercise 6.

\[ V(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{h(x_0)}{(x_0 - x)^2 + y^2} \, dx_0 \]

Now that the solution is known, change back to \( U \).

\[ \frac{\partial U}{\partial y} = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{h(x_0)}{(x_0 - x)^2 + y^2} \, dx_0 \]

Integrate both sides partially with respect to \( y \).

\[
U(x, y) = \int_{y}^{\infty} \left[ \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{h(x_0)}{(x_0 - x)^2 + s^2} \, dx_0 \right] \, ds + F(x)
\]

\[ = \frac{1}{\pi} \int_{-\infty}^{\infty} \left[ \int_{y}^{\infty} \frac{s}{(x_0 - x)^2 + s^2} \, ds \right] h(x_0) \, dx_0 + F(x) \]

Let

\[ w = (x_0 - x)^2 + s^2 \]

\[ dw = 2s \, ds \quad \rightarrow \quad \frac{dw}{2} = s \, ds. \]

Consequently,

\[
U(x, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \left[ \int_{(x_0 - x)^2 + y^2}^{\infty} \frac{1}{w} \, \frac{dw}{2} \right] h(x_0) \, dx_0 + F(x)
\]

\[ = \frac{1}{\pi} \int_{-\infty}^{\infty} \left\{ \frac{1}{2} \ln[(x_0 - x)^2 + y^2] \right\} h(x_0) \, dx_0 + F(x) \]

\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(x_0) \ln[(x_0 - x)^2 + y^2] \, dx_0 + F(x). \]