Exercise 23

Solve the Neumann problem in the quarter-plane $\{x > 0,\ y > 0\}$.

Solution

The Poisson equation will be solved in a quarter-plane D with Neumann boundary conditions.

$$\begin{aligned}
&\Delta U = f(x,y), \quad x > 0,\ y > 0 \\
&\frac{\partial U}{\partial x}(0,y) = g(y) \\
&\frac{\partial U}{\partial y}(x,0) = h(x)
\end{aligned}$$

A Neumann function representation for the solution can be obtained from Green's second identity,

$$\iint_D (u\Delta v - v\Delta u)\,dA = \int_{\text{bdy } D} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) ds.$$

Let $u = U(x,y)$ and let $v = N = N(x,y;x_0,y_0)$ be the Neumann function.

$$\iint_D (U\Delta N - N\Delta U)\,dA = \int_{\text{bdy } D} \left(U\frac{\partial N}{\partial n} - N\frac{\partial U}{\partial n}\right) ds$$

If we require N to satisfy

$$\begin{aligned}
&\Delta N = \delta(x - x_0)\delta(y - y_0), \quad x > 0,\ y > 0 \\
&\frac{\partial N}{\partial n} = c \quad \text{on bdy } D,
\end{aligned}$$

where c is a constant and (x_0, y_0) is a point in the upper half-plane, then the identity becomes

$$\iint_D [U(x,y)\delta(x-x_0)\delta(y-y_0) - N(x,y;x_0,y_0)f(x,y)]\,dA = \int_{\text{bdy } D} \left[U(x,y)c - N(x,y;x_0,y_0)\frac{\partial U}{\partial n}\right] ds.$$

Write the normal derivative as $\partial U/\partial n = \nabla U \cdot \hat{\mathbf{n}}$ and split up the integrals.

$$\begin{aligned}
&\iint_D U(x,y)\delta(x-x_0)\delta(y-y_0)\,dA - \iint_D N(x,y;x_0,y_0)f(x,y)\,dA \\
&\qquad = c\int_{\text{bdy } D} U(x,y)\,ds - \int_{\text{bdy } D} N(x,y;x_0,y_0)\nabla U \cdot \hat{\mathbf{n}}\,ds
\end{aligned}$$

The integral involving the delta functions is $U(x_0, y_0)$.

$$U(x_0,y_0) - \iint_D N(x,y;x_0,y_0)f(x,y)\,dA = c\int_{\text{bdy } D} U(x,y)\,ds - \int_{\text{bdy } D} N(x,y;x_0,y_0)\nabla U \cdot \hat{\mathbf{n}}\,ds \quad (1)$$

Determine the constant c by setting $u = 1$ and $v = N(x, y; x_0, y_0)$ in Green's second identity.

$$\iint_D \Delta N \, dA = \int_{\text{bdy } D} \frac{\partial N}{\partial n} \, ds$$

$$\iint_D \delta(x - x_0)\delta(y - y_0) \, dA = \int_{\text{bdy } D} c \, ds$$

$$1 = c \int_{\text{bdy } D} ds$$

Solve for c.

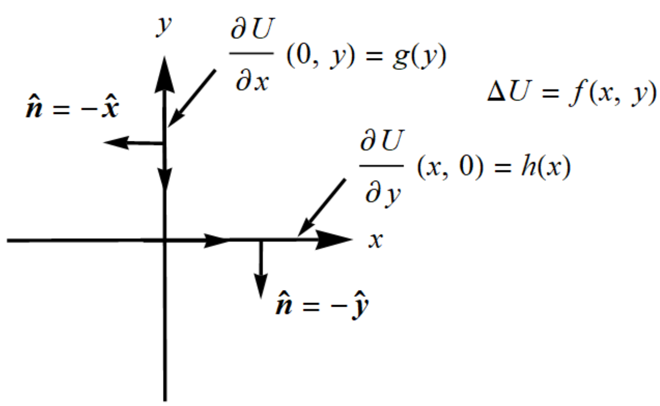
$$c = \frac{1}{\displaystyle\int_{\text{bdy } D} ds} = \frac{1}{\displaystyle\int_\infty^0 (-dy) + \int_0^\infty dx} = 0$$

Then equation (1) becomes

$$U(x_0, y_0) - \iint_D N(x, y; x_0, y_0) f(x, y) \, dA = \frac{\displaystyle\int_{\text{bdy } D} U(x, y) \, ds}{\displaystyle\int_{\text{bdy } D} ds} - \int_{\text{bdy } D} N(x, y; x_0, y_0) \nabla U \cdot \hat{\mathbf{n}} \, ds.$$

The first term on the right side is the average value of U on the boundary of the quarter-plane, a constant. Denote it as $\overline{U}$.

$$U(x_0, y_0) - \iint_D N(x, y; x_0, y_0) f(x, y) \, dA = \overline{U} - \int_{\text{bdy } D} N(x, y; x_0, y_0) \nabla U \cdot \hat{\mathbf{n}} \, ds$$



As the figure illustrates, the outward unit vector normal to the y-axis is $\hat{\mathbf{n}} = -\hat{\mathbf{x}}$, and the outward unit vector normal to the x-axis is $\hat{\mathbf{n}} = -\hat{\mathbf{y}}$.

$$\begin{aligned} U(x_0, y_0) - \iint_D N(x, y; x_0, y_0) f(x, y) \, dA = \overline{U} &- \int_\infty^0 [N(x, y; x_0, y_0) \nabla U \cdot (-\hat{\mathbf{x}})]\bigg|_{x=0} (-dy) \\ &- \int_0^\infty [N(x, y; x_0, y_0) \nabla U \cdot (-\hat{\mathbf{y}})]\bigg|_{y=0} dx \end{aligned}$$

Evaluate the dot products.

$$U(x_0,y_0)-\iint_D N(x,y;x_0,y_0)f(x,y)\,dA=\overline{U}-\int_0^\infty\left[N(x,y;x_0,y_0)\left(-\frac{\partial U}{\partial x}\right)\right]\bigg|_{x=0}dy-\int_0^\infty\left[N(x,y;x_0,y_0)\left(-\frac{\partial U}{\partial y}\right)\right]\bigg|_{y=0}dx$$

Substitute the prescribed boundary conditions and write the integration limits on the left.

$$U(x_0,y_0)-\int_0^\infty\int_0^\infty N(x,y;x_0,y_0)f(x,y)\,dx\,dy=\overline{U}+\int_0^\infty N(x,y;x_0,y_0)|_{x=0}\,g(y)\,dy+\int_0^\infty N(x,y;x_0,y_0)|_{y=0}\,h(x)\,dx$$

Solve for U.

$$U(x_0,y_0)=\overline{U}+\int_0^\infty\int_0^\infty N(x,y;x_0,y_0)f(x,y)\,dx\,dy+\int_0^\infty N(x,y;x_0,y_0)|_{x=0}\,g(y)\,dy+\int_0^\infty N(x,y;x_0,y_0)|_{y=0}\,h(x)\,dx$$

Switch the roles of x and y with those of x_0 and y_0, respectively.

$$U(x,y)=\overline{U}+\int_0^\infty\int_0^\infty N(x_0,y_0;x,y)f(x_0,y_0)\,dx_0\,dy_0+\int_0^\infty N(x_0,y_0;x,y)|_{x_0=0}\,g(y_0)\,dy_0+\int_0^\infty N(x_0,y_0;x,y)|_{y_0=0}\,h(x_0)\,dx_0 \quad (2)$$

The Neumann function will now be shown to be symmetric if $c=0$. Set $u=N(x,y;x_1,y_1)$ and $v=N(x,y;x_2,y_2)$ in Green's second identity,

$$\iint_D[N(x,y;x_1,y_1)\Delta N(x,y;x_2,y_2)-N(x,y;x_2,y_2)\Delta N(x,y;x_1,y_1)]\,dA=\int_{\text{bdy }D}\left[N(x,y;x_1,y_1)\frac{\partial N}{\partial n}(x,y;x_2,y_2)-N(x,y;x_2,y_2)\frac{\partial N}{\partial n}(x,y;x_1,y_1)\right]ds,$$

where (x_1,y_1) and (x_2,y_2) are points in D, and $N(x,y;x_1,y_1)$ and $N(x,y;x_2,y_2)$ satisfy

$$\Delta N=\delta(x-x_1)\delta(y-y_1) \quad \text{in } D \qquad\qquad \Delta N=\delta(x-x_2)\delta(y-y_2) \quad \text{in } D$$
$$\frac{\partial N}{\partial n}(x,y;x_1,y_1)=c \quad \text{on bdy } D \qquad\qquad \frac{\partial N}{\partial n}(x,y;x_2,y_2)=c \quad \text{on bdy } D.$$

Substitute these results into the identity.

$$\iint_D[N(x,y;x_1,y_1)\delta(x-x_2)\delta(y-y_2)-N(x,y;x_2,y_2)\delta(x-x_1)\delta(y-y_1)]\,dA=\int_{\text{bdy }D}[N(x,y;x_1,y_1)c-N(x,y;x_2,y_2)c]\,ds$$

Split up the integrals on the left and bring c in front of the integral on the right.

$$\iint_D N(x,y;x_1,y_1)\delta(x-x_2)\delta(y-y_2)\,dA - \iint_D N(x,y;x_2,y_2)\delta(x-x_1)\delta(y-y_1)\,dA$$
$$= c\int_{\text{bdy } D} [N(x,y;x_1,y_1) - N(x,y;x_2,y_2)]\,ds$$

If $c=0$, then the right side is zero. Evaluate the integrals on the left.

$$N(x_2,y_2;x_1,y_1) - N(x_1,y_1;x_2,y_2) = 0$$

Therefore, $N(x_2,y_2;x_1,y_1) = N(x_1,y_1;x_2,y_2)$, and equation (2) becomes

$$U(x,y) = \overline{U} + \int_0^\infty \int_0^\infty N(x,y;x_0,y_0)f(x_0,y_0)\,dx_0\,dy_0 + \int_0^\infty N(x,y;x_0,y_0)\big|_{x_0=0}\, g(y_0)\,dy_0$$
$$+ \int_0^\infty N(x,y;x_0,y_0)\big|_{y_0=0}\, h(x_0)\,dx_0.$$

The solution for Poisson's equation is known, then, if the Neumann function in the quarter-plane can be determined. Begin by finding the Neumann function in the whole plane (no boundaries).

$$\Delta\mathscr{N} = \delta(x-x_0)\delta(y-y_0), \qquad -\infty < x < \infty,\ -\infty < y < \infty$$

$\mathscr{N}$ can be interpreted as the electrostatic potential, and $\delta(x-x_0)\delta(y-y_0)$ can be interpreted as the charge density for a unit charge located at (x_0,y_0). Since there are no boundaries, $\mathscr{N}$ is expected to vary solely as a function of the radial distance from (x_0,y_0): $\mathscr{N} = \mathscr{N}(r)$, where $r = \sqrt{(x-x_0)^2 + (y-y_0)^2}$. Integrate both sides over a disk centered at (x_0,y_0) with radius r.

$$\iint_{(x-x_0)^2+(y-y_0)^2\le r^2} \Delta\mathscr{N}\,dA = \iint_{(x-x_0)^2+(y-y_0)^2\le r^2} \delta(x-x_0)\delta(y-y_0)\,dA$$

Since the disk contains (x_0,y_0), the right side is 1. Write the Laplacian operator as $\Delta = \nabla^2$

$$\iint_{(x-x_0)^2+(y-y_0)^2\le r^2} \nabla^2\mathscr{N}\,dA = 1$$

$$\iint_{(x-x_0)^2+(y-y_0)^2\le r^2} \nabla\cdot\nabla\mathscr{N}\,dA = 1$$

and apply the two-dimensional divergence theorem.

$$\int_{(x-x_0)^2+(y-y_0)^2= r^2} \nabla\mathscr{N}\cdot\hat{\mathbf{r}}\,ds = 1$$

Here $\hat{\mathbf{r}}$ is the unit vector normal to this disk at every point on the boundary.

$$\int_{(x-x_0)^2+(y-y_0)^2= r^2} \frac{d\mathscr{N}}{dr}\,ds = 1$$

Because $\mathscr{N}$ only depends on $\imath$, its derivative is constant on the disk's boundary.

$$\frac{d\mathscr{N}}{d\imath}\int\limits_{(x-x_0)^2+(y-y_0)^2=\imath^2} ds = 1$$

This line integral is just the disk's circumference.

$$\frac{d\mathscr{N}}{d\imath}(2\pi\imath) = 1$$

Divide both sides by $2\pi\imath$.

$$\frac{d\mathscr{N}}{d\imath} = \frac{1}{2\pi\imath}$$

Integrate both sides with respect to $\imath$.

$$\mathscr{N}(\imath) = \frac{1}{2\pi}\ln\imath$$

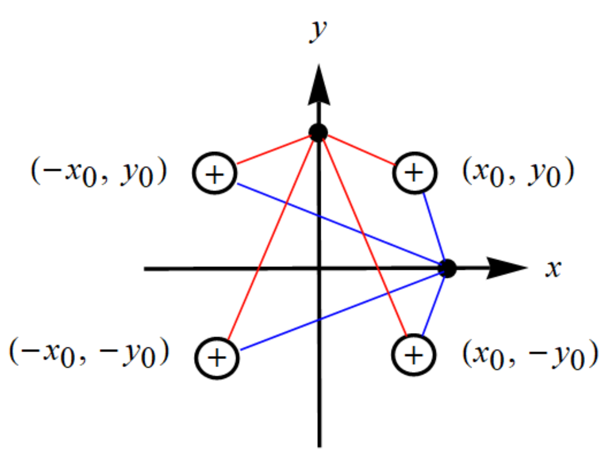
The Neumann function for the whole plane is then

$$\mathscr{N}(x, y; x_0, y_0) = \frac{1}{2\pi}\ln\sqrt{(x-x_0)^2+(y-y_0)^2}.$$

Now that it's known, the Neumann function for the quarter-plane can be determined by the method of images. A convocation of point charges in the whole plane will be arranged so that the boundary condition,

$$\frac{\partial N}{\partial n} = c = \frac{1}{\displaystyle\int\limits_{\text{bdy } D} ds} = \frac{1}{\displaystyle\int_\infty^0 (-dy) + \int_0^\infty dx} = 0 \text{ on bdy } D \quad \Rightarrow \quad \begin{cases} \dfrac{\partial N}{\partial x}(0, y) = 0 \\ \dfrac{\partial N}{\partial y}(x, 0) = 0 \end{cases},$$

is satisfied. These derivatives of potential with respect to x and y can be interpreted as the x- and y-components of the electric field, respectively. For a positive unit charge at (x_0, y_0), place positive unit charges at $(x_0, -y_0)$, $(-x_0, y_0)$, and $(-x_0, -y_0)$ so that every point on the positive x- and y-axes is equally spaced from each pair.



The quarter-plane Neumann function can now be written.

$$N(x,y;x_0,y_0) = +\mathscr{N}(x,y;x_0,y_0) + \mathscr{N}(x,y;x_0,-y_0) + \mathscr{N}(x,y;-x_0,y_0) + \mathscr{N}(x,y;-x_0,-y_0), \quad x>0,\ y>0$$

Because $\mathscr{N}$ is defined over the whole plane, it's important to note the restriction to $x>0$, $y>0$ for N.

$$\begin{aligned}
N(x,y;x_0,y_0) &= \frac{1}{2\pi}\ln\sqrt{(x-x_0)^2+(y-y_0)^2} + \frac{1}{2\pi}\ln\sqrt{(x-x_0)^2+(y+y_0)^2} + \frac{1}{2\pi}\ln\sqrt{(x+x_0)^2+(y-y_0)^2} + \frac{1}{2\pi}\ln\sqrt{(x+x_0)^2+(y+y_0)^2} \\
&= \frac{1}{2\pi}\ln\left[\sqrt{(x-x_0)^2+(y-y_0)^2}\sqrt{(x-x_0)^2+(y+y_0)^2}\sqrt{(x+x_0)^2+(y-y_0)^2}\sqrt{(x+x_0)^2+(y+y_0)^2}\right] \\
&= \frac{1}{4\pi}\ln\left\{[(x-x_0)^2+(y-y_0)^2][(x-x_0)^2+(y+y_0)^2][(x+x_0)^2+(y-y_0)^2][(x+x_0)^2+(y+y_0)^2]\right\}
\end{aligned}$$

Now set $x_0=0$ and $y_0=0$ in this result.

$$\begin{aligned}
N(x,y;x_0,y_0)|_{x_0=0} &= \frac{1}{4\pi}\ln\left\{[x^2+(y-y_0)^2][x^2+(y+y_0)^2][x^2+(y-y_0)^2][x^2+(y+y_0)^2]\right\} \\
&= \frac{1}{4\pi}\ln\left\{[x^2+(y-y_0)^2]^2[x^2+(y+y_0)^2]^2\right\} \\
&= \frac{1}{2\pi}\ln\left\{[x^2+(y-y_0)^2][x^2+(y+y_0)^2]\right\}
\end{aligned}$$

$$\begin{aligned}
N(x,y;x_0,y_0)|_{y_0=0} &= \frac{1}{4\pi}\ln\left\{[(x-x_0)^2+y^2][(x-x_0)^2+y^2][(x+x_0)^2+y^2][(x+x_0)^2+y^2]\right\} \\
&= \frac{1}{4\pi}\ln\left\{[(x-x_0)^2+y^2]^2[(x+x_0)^2+y^2]^2\right\} \\
&= \frac{1}{2\pi}\ln\left\{[(x-x_0)^2+y^2][(x+x_0)^2+y^2]\right\}
\end{aligned}$$

Therefore, the solution to Poisson's equation in the quarter-plane $(x>0,\ y>0)$ with Neumann boundary conditions is

$$\begin{aligned}
U(x,y) = \overline{U} + \frac{1}{4\pi}\int_0^\infty\int_0^\infty f(x_0,y_0)\ln\left\{[(x-x_0)^2+(y-y_0)^2][(x-x_0)^2+(y+y_0)^2][(x+x_0)^2+(y-y_0)^2][(x+x_0)^2+(y+y_0)^2]\right\}dx_0\,dy_0 \\
+ \frac{1}{2\pi}\int_0^\infty g(y_0)\ln\left\{[x^2+(y-y_0)^2][x^2+(y+y_0)^2]\right\}dy_0 + \frac{1}{2\pi}\int_0^\infty h(x_0)\ln\left\{[(x-x_0)^2+y^2][(x+x_0)^2+y^2]\right\}dx_0.
\end{aligned}$$