Exercise 4

(*Lorentz invariance of the wave equation*) Thinking of the coordinates of space-time as 4-vectors (x, y, z, t), let Γ be the diagonal matrix with the diagonal entries $1, 1, 1, -1$. Another matrix L is called a *Lorentz transformation* if L has an inverse and $L^{-1} = \Gamma\, {}^tL\,\Gamma$, where ${}^tL$ is the transpose.

(a) If L and M are Lorentz, show that LM and L^{-1} also are.

(b) Show that L is Lorentz if and only if $m(L\mathbf{v}) = m(\mathbf{v})$ for all 4-vectors $\mathbf{v} = (x, y, z, t)$, where $m(\mathbf{v}) = x^2 + y^2 + z^2 - t^2$ is called the *Lorentz metric*.

(c) If $u(x, y, z, t)$ is any function and L is Lorentz, let $U(x, y, z, t) = u(L(x, y, z, t))$. Show that

$$U_{xx} + U_{yy} + U_{zz} - U_{tt} = u_{xx} + u_{yy} + u_{zz} - u_{tt}.$$

(d) Explain the meaning of a Lorentz transformation in more geometrical terms. (*Hint:* Consider the level sets of $m(\mathbf{v})$.)

Solution

Start by writing the matrix of Γ.

$$\Gamma = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Notice that Γ is its own inverse and its own transpose.

$$\Gamma = \Gamma^{-1} = {}^t\Gamma$$

Part (a)

Suppose L and M are Lorentz. Then, by the given definition,

$$L^{-1} = \Gamma\, {}^tL\,\Gamma \quad \text{and} \quad M^{-1} = \Gamma\, {}^tM\,\Gamma$$

In order to show that LM is Lorentz, multiply M^{-1} and L^{-1} together.

$$M^{-1}L^{-1} = (\Gamma\, {}^tM\,\Gamma)(\Gamma\, {}^tL\,\Gamma)$$

Use one of the basic properties of the inverse on the left side: $(LM)^{-1} = M^{-1}L^{-1}$. Use the fact that matrix multiplication is associative on the right side.

$$(LM)^{-1} = \Gamma\, {}^tM\,(\Gamma\Gamma)\, {}^tL\,\Gamma$$

$\Gamma\Gamma$ is the identity matrix.

$$(LM)^{-1} = \Gamma\, {}^tM\, {}^tL\,\Gamma$$

Use one of the basic properties of the transpose: ${}^t(LM) = {}^tM\, {}^tL$.

$$(LM)^{-1} = \Gamma\, {}^t(LM)\,\Gamma$$

Therefore, LM is a Lorentz transformation.

Now it will be shown that L^{-1} is Lorentz.

$$\begin{aligned} L^{-1} &= \Gamma\,{}^tL\,\Gamma \\ &= (\Gamma\,{}^tL)\,\Gamma \end{aligned}$$

Take the transpose of both sides and apply its basic property twice.

$$\begin{aligned} {}^t(L^{-1}) &= {}^t[(\Gamma\,{}^tL)\,\Gamma] \\ &= {}^t\Gamma\,{}^t(\Gamma\,{}^tL) \\ &= {}^t\Gamma\,{}^t({}^tL)\,{}^t\Gamma \end{aligned}$$

Note that ${}^t\Gamma = \Gamma$ and ${}^t({}^tL) = L$.

$${}^t(L^{-1}) = \Gamma\,L\,\Gamma$$

Premultiply both sides by Γ.

$$\begin{aligned} \Gamma\,{}^t(L^{-1}) &= \Gamma(\Gamma\,L\,\Gamma) \\ &= (\Gamma\Gamma)\,L\,\Gamma \\ &= L\,\Gamma \end{aligned}$$

Postmultiply both sides by Γ.

$$\begin{aligned} \Gamma\,{}^t(L^{-1})\,\Gamma &= (L\,\Gamma)\Gamma \\ &= L(\Gamma\Gamma) \\ &= L \end{aligned}$$

Since $L = (L^{-1})^{-1}$, we have

$$(L^{-1})^{-1} = \Gamma\,{}^t(L^{-1})\,\Gamma.$$

Therefore, L^{-1} is a Lorentz transformation.

Part (b)

Suppose that L is Lorentz and let $x_1 = x$, $x_2 = y$, $x_3 = z$, and $x_4 = t$ so that series can be used. Then $\mathbf{v} = (x_1, x_2, x_3, x_4)$. The aim is to show that $m(\mathbf{v}) = m(L\mathbf{v})$.

$$\begin{aligned} m(\mathbf{v}) &= x^2 + y^2 + z^2 - t^2 \\ &= \sum_{i=1}^{4}\sum_{j=1}^{4} \Gamma_{ij} x_i x_j \end{aligned}$$

Because L is Lorentz, it's true that $L^{-1} = \Gamma\,{}^tL\,\Gamma$. Premultiply both sides by Γ and postmultiply both sides by L to obtain $\Gamma L^{-1} L = \Gamma\Gamma\,{}^tL\Gamma L$, or $\Gamma = {}^tL\,\Gamma\,L$.

$$= \sum_{i=1}^{4}\sum_{j=1}^{4} ({}^tL\,\Gamma\,L)_{ij} x_i x_j$$

In parentheses is the product of three 4×4 matrices. To get the ij-component, i and j will be the leftmost and rightmost indices, and the indices of summation will be connected in between them.

$$m(\mathbf{v}) = \sum_{i=1}^{4} \sum_{j=1}^{4} \left[\sum_{k=1}^{4} \sum_{l=1}^{4} ({}^{t}L)_{ik} \Gamma_{kl} L_{lj} \right] x_i x_j$$

The transpose has the effect of switching the indices.

$$= \sum_{i=1}^{4} \sum_{j=1}^{4} \left(\sum_{k=1}^{4} \sum_{l=1}^{4} L_{ki} \Gamma_{kl} L_{lj} \right) x_i x_j$$

The limits of summation are constant, so the sums can be arranged however we like.

$$= \sum_{k=1}^{4} \sum_{l=1}^{4} \Gamma_{kl} \left(\sum_{i=1}^{4} L_{ki} x_i \right) \left(\sum_{j=1}^{4} L_{lj} x_j \right)$$

As L is a 4×4 matrix and $\mathbf{v}$ is a 4×1 matrix, the terms in parentheses are the k- and l-components of $L\mathbf{v}$.

$$\begin{aligned} &= \sum_{k=1}^{4} \sum_{l=1}^{4} \Gamma_{kl} \left(L\mathbf{v} \right)_k \left(L\mathbf{v} \right)_l \\ &= m(L\mathbf{v}) \end{aligned}$$

Now that it has been shown that $m(\mathbf{v}) = m(L\mathbf{v})$, the first part of the proof is complete. For the second part, suppose that $m(\mathbf{v}) = m(L\mathbf{v})$. The aim is to show that L is Lorentz. We have

$$\begin{aligned} m(\mathbf{v}) &= x^2 + y^2 + z^2 - t^2 \\ &= \sum_{i=1}^{4} \sum_{j=1}^{4} \Gamma_{ij} x_i x_j \end{aligned}$$

and (working backwards)

$$\begin{aligned} m(L\mathbf{v}) &= \sum_{k=1}^{4} \sum_{l=1}^{4} \Gamma_{kl} \left(L\mathbf{v} \right)_k \left(L\mathbf{v} \right)_l \\ &= \sum_{k=1}^{4} \sum_{l=1}^{4} \Gamma_{kl} \left(\sum_{i=1}^{4} L_{ki} x_i \right) \left(\sum_{j=1}^{4} L_{lj} x_j \right) \\ &= \sum_{i=1}^{4} \sum_{j=1}^{4} \left(\sum_{k=1}^{4} \sum_{l=1}^{4} L_{ki} \Gamma_{kl} L_{lj} \right) x_i x_j \\ &= \sum_{i=1}^{4} \sum_{j=1}^{4} \left[\sum_{k=1}^{4} \sum_{l=1}^{4} ({}^{t}L)_{ik} \Gamma_{kl} L_{lj} \right] x_i x_j \\ &= \sum_{i=1}^{4} \sum_{j=1}^{4} ({}^{t}L\, \Gamma\, L)_{ij} x_i x_j. \end{aligned}$$

Because $m(\mathbf{v}) = m(L\mathbf{v})$,

$$\sum_{i=1}^{4}\sum_{j=1}^{4}\Gamma_{ij}x_ix_j = \sum_{i=1}^{4}\sum_{j=1}^{4}({}^tL\,\Gamma\,L)_{ij}x_ix_j.$$

This implies that

$$\Gamma = {}^tL\,\Gamma\,L.$$

Premultiply both sides by Γ and postmultiply both sides by L^{-1}.

$$\Gamma\Gamma L^{-1} = \Gamma\,{}^tL\,\Gamma\,LL^{-1}$$

LL^{-1} and $\Gamma\Gamma$ are equal to the identity matrix, so

$$L^{-1} = \Gamma\,{}^tL\,\Gamma,$$

which means L is Lorentz. Therefore, L is Lorentz if and only if $m(L\mathbf{v}) = m(\mathbf{v})$ for all 4-vectors $\mathbf{v} = (x, y, z, t)$.

Part (c)

Let $x_1 = x$, $x_2 = y$, $x_3 = z$, and $x_4 = t$ so that series can be used. Then $\mathbf{v} = (x_1, x_2, x_3, x_4)$.

$$u_{xx} + u_{yy} + u_{zz} - u_{tt} = \sum_{i=1}^{4}\sum_{j=1}^{4}\Gamma_{ij}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}u(L\mathbf{v})$$

Apply the chain rule to differentiate $u(L\mathbf{v})$.

$$= \sum_{i=1}^{4}\sum_{j=1}^{4}\Gamma_{ij}\left[\sum_{k=1}^{4}\sum_{l=1}^{4}\frac{\partial^2 u(L\mathbf{v})}{\partial x_k\partial x_l}\frac{\partial}{\partial x_i}(L\mathbf{v})_k\frac{\partial}{\partial x_j}(L\mathbf{v})_l\right]$$

Replace $u(L\mathbf{v})$ with $U(\mathbf{v}) = U$ and write the expression for the multiplication of 4×4 matrix L with 4×1 matrix $\mathbf{v}$.

$$= \sum_{i=1}^{4}\sum_{j=1}^{4}\Gamma_{ij}\left[\sum_{k=1}^{4}\sum_{l=1}^{4}\frac{\partial^2 U}{\partial x_k\partial x_l}\frac{\partial}{\partial x_i}\left(\sum_{m=1}^{4}L_{km}x_m\right)\frac{\partial}{\partial x_j}\left(\sum_{m=1}^{4}L_{lm}x_m\right)\right]$$

The derivative of a sum is the sum of the derivatives.

$$= \sum_{i=1}^{4}\sum_{j=1}^{4}\Gamma_{ij}\left[\sum_{k=1}^{4}\sum_{l=1}^{4}\frac{\partial^2 U}{\partial x_k\partial x_l}\left(\sum_{m=1}^{4}L_{km}\frac{\partial x_m}{\partial x_i}\right)\left(\sum_{m=1}^{4}L_{lm}\frac{\partial x_m}{\partial x_j}\right)\right]$$

Only if m is equal to i in the first sum will $\partial x_m/\partial x_i$ be equal to 1. Similarly, only if m is equal to j in the second sum will $\partial x_m/\partial x_j$ be equal to 1. The Kronecker delta symbol represents this.

$$= \sum_{i=1}^{4}\sum_{j=1}^{4}\Gamma_{ij}\left[\sum_{k=1}^{4}\sum_{l=1}^{4}\frac{\partial^2 U}{\partial x_k\partial x_l}\left(\sum_{m=1}^{4}L_{km}\delta_{mi}\right)\left(\sum_{m=1}^{4}L_{lm}\delta_{mj}\right)\right]$$

δ_{mi} sifts $m = i$ from the first summand in parentheses, and δ_{mj} sifts $m = j$ from the second one.

$$u_{xx} + u_{yy} + u_{zz} - u_{tt} = \sum_{i=1}^{4}\sum_{j=1}^{4}\Gamma_{ij}\left[\sum_{k=1}^{4}\sum_{l=1}^{4}\frac{\partial^2 U}{\partial x_k \partial x_l}(L_{ki})(L_{lj})\right]$$

The limits of summation are constant, so the sums can be arranged however we like.

$$= \sum_{k=1}^{4}\sum_{l=1}^{4}\left(\sum_{i=1}^{4}\sum_{j=1}^{4}L_{ki}\Gamma_{ij}L_{lj}\right)\frac{\partial^2 U}{\partial x_k \partial x_l}$$

In order to connect the inner indices, use the transpose of L.

$$= \sum_{k=1}^{4}\sum_{l=1}^{4}\left[\sum_{i=1}^{4}\sum_{j=1}^{4}L_{ki}\Gamma_{ij}({}^tL)_{jl}\right]\frac{\partial^2 U(\mathbf{v})}{\partial x_k \partial x_l}$$

Because L is Lorentz, it's true that $L^{-1} = \Gamma\,{}^tL\Gamma$. Premultiply both sides by L and postmultiply both sides by Γ to obtain $LL^{-1}\Gamma = L\,\Gamma\,{}^tL\Gamma\Gamma$, or $\Gamma = L\,\Gamma\,{}^tL\cdot$ Consequently, the term in square brackets is the kl-component of the matrix product $L\,\Gamma\,{}^tL$, which is equal to Γ.

$$\begin{aligned} &= \sum_{k=1}^{4}\sum_{l=1}^{4}\Gamma_{kl}\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_l}U(\mathbf{v}) \\ &= U_{xx} + U_{yy} + U_{zz} - U_{tt} \end{aligned}$$

Therefore, the wave equation is invariant under Lorentz transformations.

$$\boxed{u_{xx} + u_{yy} + u_{zz} - u_{tt} = U_{xx} + U_{yy} + U_{zz} - U_{tt}}$$

Part (d)

The Lorentz metric of a 4-vector $m(\mathbf{v})$ can be written as follows.

$$\begin{aligned} m(\mathbf{v}) &= x^2 + y^2 + z^2 - t^2 \\ &= (x, y, z, it)\cdot(x, y, z, it) \\ &= |(x, y, z, it)|^2 \end{aligned}$$

We see from this form that $m(\mathbf{v})$ represents the magnitude of $\mathbf{v} = (x, y, z, t)$ in space-time. Unlike the magnitude of a regular 3-vector $\mathbf{x} = (x, y, z)$, which is $|\mathbf{x}|^2 = x^2 + y^2 + z^2$, it can be negative. The magnitude of $\mathbf{x}$ only depends on its distance from the origin. That is, it is invariant under rotations in space: $|R\mathbf{x}| = |\mathbf{x}|$ for any rotation matrix R. By comparison, since $m(L\mathbf{v}) = m(\mathbf{v})$, L can be thought of as an imaginary rotation in space-time. In the theory of special relativity, for example, it is a Lorentz transformation that relates the space-time coordinates of two observers moving relative to one another at constant velocity. The extent of rotation that occurs in space-time is related to the observers' relative velocity.