Exercise 4

(*Lorentz invariance of the wave equation*) Thinking of the coordinates of space-time as 4-vectors (x, y, z, t), let Γ be the diagonal matrix with the diagonal entries $1, 1, 1, -1$. Another matrix L is called a *Lorentz transformation* if L has an inverse and $L^{-1} = \Gamma\, {}^tL\, \Gamma$, where ${}^tL$ is the transpose.

(a) If L and M are Lorentz, show that LM and L^{-1} also are.

(b) Show that L is Lorentz if and only if $m(L\mathbf{v}) = m(\mathbf{v})$ for all 4-vectors $\mathbf{v} = (x, y, z, t)$, where $m(\mathbf{v}) = x^2 + y^2 + z^2 - t^2$ is called the *Lorentz metric*.

(c) If $u(x, y, z, t)$ is any function and L is Lorentz, let $U(x, y, z, t) = u(L(x, y, z, t))$. Show that

$$U_{xx} + U_{yy} + U_{zz} - U_{tt} = u_{xx} + u_{yy} + u_{zz} - u_{tt}.$$

(d) Explain the meaning of a Lorentz transformation in more geometrical terms. (*Hint:* Consider the level sets of $m(\mathbf{v})$.)