Exercise 2

Verify that $(c^2t^2 - x^2 - y^2 - z^2)^{-1}$ satisfies the wave equation except on the light cone.

Solution

The three-dimensional wave equation is

$$u_{tt} = c^2(u_{xx} + u_{yy} + u_{zz}).$$

Find all the second derivatives of $u(x, y, z, t) = (c^2t^2 - x^2 - y^2 - z^2)^{-1}$.

$$\begin{aligned}
u_t &= -(c^2t^2 - x^2 - y^2 - z^2)^{-2}(2c^2t) \\
u_{tt} &= 2(c^2t^2 - x^2 - y^2 - z^2)^{-3}(2c^2t)^2 - (c^2t^2 - x^2 - y^2 - z^2)^{-2}(2c^2) \\
&= \frac{8c^4t^2}{(c^2t^2 - x^2 - y^2 - z^2)^3} - \frac{2c^2}{(c^2t^2 - x^2 - y^2 - z^2)^2} \\
&= \frac{8c^4t^2 - 2c^2(c^2t^2 - x^2 - y^2 - z^2)}{(c^2t^2 - x^2 - y^2 - z^2)^3} \\
&= \frac{6c^4t^2 + 2c^2(x^2 + y^2 + z^2)}{(c^2t^2 - x^2 - y^2 - z^2)^3} \\
u_x &= -(c^2t^2 - x^2 - y^2 - z^2)^{-2}(-2x) \\
u_{xx} &= 2(c^2t^2 - x^2 - y^2 - z^2)^{-3}(-2x)^2 - (c^2t^2 - x^2 - y^2 - z^2)^{-2}(-2) \\
&= \frac{8x^2}{(c^2t^2 - x^2 - y^2 - z^2)^3} + \frac{2}{(c^2t^2 - x^2 - y^2 - z^2)^2} \\
&= \frac{8x^2 + 2(c^2t^2 - x^2 - y^2 - z^2)}{(c^2t^2 - x^2 - y^2 - z^2)^3} \\
&= \frac{6x^2 + 2(c^2t^2 - y^2 - z^2)}{(c^2t^2 - x^2 - y^2 - z^2)^3} \\
u_y &= -(c^2t^2 - x^2 - y^2 - z^2)^{-2}(-2y) \\
u_{yy} &= 2(c^2t^2 - x^2 - y^2 - z^2)^{-3}(-2y)^2 - (c^2t^2 - x^2 - y^2 - z^2)^{-2}(-2) \\
&= \frac{8y^2}{(c^2t^2 - x^2 - y^2 - z^2)^3} + \frac{2}{(c^2t^2 - x^2 - y^2 - z^2)^2} \\
&= \frac{8y^2 + 2(c^2t^2 - x^2 - y^2 - z^2)}{(c^2t^2 - x^2 - y^2 - z^2)^3} \\
&= \frac{6y^2 + 2(c^2t^2 - x^2 - z^2)}{(c^2t^2 - x^2 - y^2 - z^2)^3} \\
u_z &= -(c^2t^2 - x^2 - y^2 - z^2)^{-2}(-2z) \\
u_{zz} &= 2(c^2t^2 - x^2 - y^2 - z^2)^{-3}(-2z)^2 - (c^2t^2 - x^2 - y^2 - z^2)^{-2}(-2) \\
&= \frac{8z^2}{(c^2t^2 - x^2 - y^2 - z^2)^3} + \frac{2}{(c^2t^2 - x^2 - y^2 - z^2)^2} \\
&= \frac{8z^2 + 2(c^2t^2 - x^2 - y^2 - z^2)}{(c^2t^2 - x^2 - y^2 - z^2)^3} \\
&= \frac{6z^2 + 2(c^2t^2 - x^2 - y^2)}{(c^2t^2 - x^2 - y^2 - z^2)^3}
\end{aligned}$$

Substitute these formulas into the wave equation to check whether $u(x, y, z, t)$ is indeed a solution.

$$\begin{aligned}
\frac{6c^4t^2 + 2c^2(x^2+y^2+z^2)}{(c^2t^2-x^2-y^2-z^2)^3} &\stackrel{?}{=} c^2\left[\frac{6x^2+2(c^2t^2-y^2-z^2)}{(c^2t^2-x^2-y^2-z^2)^3} + \frac{6y^2+2(c^2t^2-x^2-z^2)}{(c^2t^2-x^2-y^2-z^2)^3} + \frac{6z^2+2(c^2t^2-x^2-y^2)}{(c^2t^2-x^2-y^2-z^2)^3}\right] \\
&\stackrel{?}{=} c^2\left[\frac{6x^2+2(c^2t^2-y^2-z^2)+6y^2+2(c^2t^2-x^2-z^2)+6z^2+2(c^2t^2-x^2-y^2)}{(c^2t^2-x^2-y^2-z^2)^3}\right] \\
&\stackrel{?}{=} c^2\left[\frac{2x^2+2y^2+2z^2+6c^2t^2}{(c^2t^2-x^2-y^2-z^2)^3}\right] \\
&= \frac{6c^4t^2 + 2c^2(x^2+y^2+z^2)}{(c^2t^2-x^2-y^2-z^2)^3}
\end{aligned}$$

Therefore, $u(x, y, z, t) = (c^2t^2 - x^2 - y^2 - z^2)^{-1}$ satisfies the wave equation except when $c^2t^2 = x^2 + y^2 + z^2$.