Exercise 8

Consider the equation $u_{tt} - c^2\Delta u + m^2 u = 0$, where $m > 0$, known as the *Klein–Gordon equation*.

(a) What is the energy? Show that it is a constant.

(b) Prove the causality principle for it.

Solution

Part (a)

Expand the Laplacian operator in Cartesian coordinates for the time being.

$$u_{tt} - c^2(u_{xx} + u_{yy} + u_{zz}) + m^2 u = 0$$

Multiply both sides by u_t.

$$u_t u_{tt} - c^2(u_t u_{xx} + u_t u_{yy} + u_t u_{zz}) + m^2 u u_t = 0$$

Rewrite each term in this equation.

$$\frac{1}{2}\frac{\partial}{\partial t}(u_t^2) - c^2\left[\frac{\partial}{\partial x}(u_t u_x) - u_{xt}u_x + \frac{\partial}{\partial y}(u_t u_y) - u_{yt}u_y + \frac{\partial}{\partial z}(u_t u_z) - u_{zt}u_z\right] + \frac{m^2}{2}\frac{\partial}{\partial t}(u^2) = 0$$

$$\frac{1}{2}\frac{\partial}{\partial t}(u_t^2) - c^2\left[\frac{\partial}{\partial x}(u_t u_x) - \frac{1}{2}\frac{\partial}{\partial t}(u_x^2) + \frac{\partial}{\partial y}(u_t u_y) - \frac{1}{2}\frac{\partial}{\partial t}(u_y^2) + \frac{\partial}{\partial z}(u_t u_z) - \frac{1}{2}\frac{\partial}{\partial t}(u_z^2)\right] + \frac{m^2}{2}\frac{\partial}{\partial t}(u^2) = 0$$

$$\frac{1}{2}\frac{\partial}{\partial t}\left[u_t^2 + m^2u^2 + c^2(u_x^2 + u_y^2 + u_z^2)\right] - c^2\left[\frac{\partial}{\partial x}(u_t u_x) + \frac{\partial}{\partial y}(u_t u_y) + \frac{\partial}{\partial z}(u_t u_z)\right] = 0$$

Reintroduce vector operators into the equation.

$$\frac{1}{2}\frac{\partial}{\partial t}(u_t^2 + m^2u^2 + c^2|\nabla u|^2) - c^2\nabla\cdot(u_t\nabla u) = 0$$

$$\frac{\partial}{\partial t}\left[\frac{1}{2}(u_t^2 + m^2u^2 + c^2|\nabla u|^2)\right] = c^2\nabla\cdot(u_t\nabla u)$$

Integrate both sides of the equation over all of space.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\partial}{\partial t}\left[\frac{1}{2}(u_t^2 + m^2u^2 + c^2|\nabla u|^2)\right]dV = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c^2\nabla\cdot(u_t\nabla u)\,dV$$

Apply the divergence theorem on the right side to turn the volume integral into a surface integral over the boundary of space.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\partial}{\partial t}\left[\frac{1}{2}(u_t^2 + m^2u^2 + c^2|\nabla u|^2)\right]dV = \oiint_{\text{bdy space}} c^2(u_t\nabla u)\cdot\mathbf{n}\,dS$$

Assuming that u_t, u_x, u_y, and u_z tend to zero far from the origin (as $|\mathbf{x}| \to \infty$), the surface integral vanishes.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\partial}{\partial t}\left[\frac{1}{2}(u_t^2 + m^2u^2 + c^2|\nabla u|^2)\right]dV = 0$$

The volume integral wipes out the spatial variables, leaving only t. The partial derivative with respect to t in the integrand becomes a total derivative in front of the integral as a result.

$$\frac{d}{dt}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\frac{1}{2}(u_t^2+m^2u^2+c^2|\nabla u|^2)\right]dV=0$$

Integrate both sides with respect to t.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\frac{1}{2}(u_t^2+m^2u^2+c^2|\nabla u|^2)\right]dV=C_1$$

Therefore, the total energy of a solution to the Klein–Gordon equation is

$$\boxed{\mathscr{E}_{\text{total}}(t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\frac{1}{2}(u_t^2+m^2u^2+c^2|\nabla u|^2)\right]dV,}$$

and it is constant in time.

Part (b)

The principle of causality for the Klein–Gordon equation in three dimensions states that the solution u at a particular point in space-time (x_0, y_0, z_0, t_0) is completely determined by the initial conditions, $u(x,y,z,0)=\phi(x,y,z)$ and $u_t(x,y,z,0)=\psi(x,y,z)$, in the solid ball $(x-x_0)^2+(y-y_0)^2+(z-z_0)^2\leq c^2t_0^2$. Consider first the Klein–Gordon equation in one dimension.

$$u_{tt}-c^2u_{xx}+m^2u=0$$

Comparing this to the general form of a linear second-order PDE, $Au_{tt}+Bu_{tx}+Cu_{xx}+Du_x+Eu_y+Fu=G$, we see that $A=1$, $B=0$, $C=-c^2$, $D=0$, $E=0$, $F=m^2$, and $G=0$. The equations for the characteristic curves of the wave equation are given by

$$\begin{aligned}\frac{dx}{dt}&=\frac{1}{2A}(B\pm\sqrt{B^2-4AC})\\&=\frac{1}{2}(\pm\sqrt{4c^2})\\&=\pm c.\end{aligned}$$

Solving these ODEs results in two families of characteristic curves, each with its own characteristic coordinate.

$$\frac{dx}{dt}=\pm c\quad\rightarrow\quad\begin{cases}x=ct+\xi\\x=-ct+\eta\end{cases}$$

The characteristic curves in the xt-plane are lines. The equations for the ones passing through a particular point (x_0, t_0) are

$$x-x_0=\pm c(t-t_0).$$

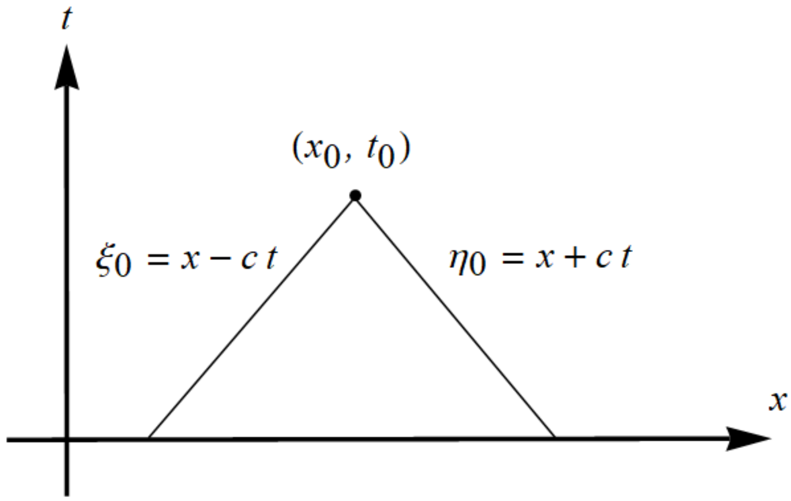

Figure 1: Each point in the xt-plane has a characteristic triangle associated with it.

Now introduce a second and third spatial dimension, y and z. Rotating the characteristic lines about the t-axis results in characteristic cones for the three-dimensional Klein–Gordon equation.

$$\begin{aligned} |\mathbf{x} - \mathbf{x}_0| &= \pm c(t - t_0) \\ \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} &= \pm c(t - t_0) \\ (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 &= c^2(t - t_0)^2 \end{aligned} \tag{1}$$

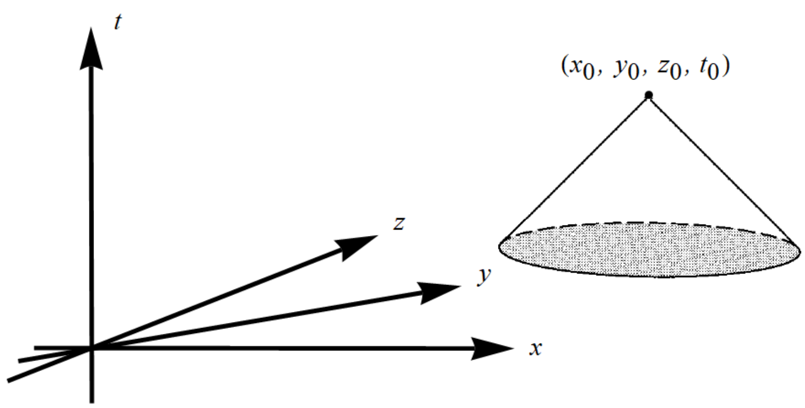

Figure 2: Each point in four-dimensional space-time has a characteristic cone associated with it. The shaded hyperdisk lies in the xyz-plane and represents a solid ball in xyz-space, that is, when $t = 0$: $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 \leq c^2 t_0^2$. The x-, y-, and z-axes are perpendicular to each other in addition to the t-axis.

The three-dimensional Klein–Gordon equation is

$$u_{tt} - c^2(u_{xx} + u_{yy} + u_{zz}) + m^2 u = 0.$$

Multiply both sides by u_t.

$$u_t u_{tt} - c^2(u_t u_{xx} + u_t u_{yy} + u_t u_{zz}) + m^2 u u_t = 0$$

Rewrite each term in the equation as follows.

$$\frac{1}{2}\frac{\partial}{\partial t}(u_t^2) - c^2\left[\frac{\partial}{\partial x}(u_t u_x) - u_{xt}u_x + \frac{\partial}{\partial y}(u_t u_y) - u_{yt}u_y + \frac{\partial}{\partial z}(u_t u_z) - u_{zt}u_z\right] + \frac{m^2}{2}\frac{\partial}{\partial t}(u^2) = 0$$

$$\frac{1}{2}\frac{\partial}{\partial t}(u_t^2) - c^2\left[\frac{\partial}{\partial x}(u_t u_x) - \frac{1}{2}\frac{\partial}{\partial t}(u_x^2) + \frac{\partial}{\partial y}(u_t u_y) - \frac{1}{2}\frac{\partial}{\partial t}(u_y^2) + \frac{\partial}{\partial z}(u_t u_z) - \frac{1}{2}\frac{\partial}{\partial t}(u_z^2)\right] + \frac{m^2}{2}\frac{\partial}{\partial t}(u^2) = 0$$

$$-c^2\left[\frac{\partial}{\partial x}(u_t u_x) + \frac{\partial}{\partial y}(u_t u_y) + \frac{\partial}{\partial z}(u_t u_z)\right] + \frac{1}{2}\frac{\partial}{\partial t}\left[u_t^2 + m^2u^2 + c^2(u_x^2 + u_y^2 + u_z^2)\right] = 0$$

$$\frac{\partial}{\partial x}(-c^2 u_t u_x) + \frac{\partial}{\partial y}(-c^2 u_t u_y) + \frac{\partial}{\partial z}(-c^2 u_t u_z) + \frac{\partial}{\partial t}\left[\frac{1}{2}(u_t^2 + m^2u^2 + c^2|\nabla u|^2)\right] = 0$$

Integrate both sides over the four-dimensional volume of the frustum W illustrated in Figure 3.

$$\iiiint_W \left\{\frac{\partial}{\partial x}(-c^2 u_t u_x) + \frac{\partial}{\partial y}(-c^2 u_t u_y) + \frac{\partial}{\partial z}(-c^2 u_t u_z) + \frac{\partial}{\partial t}\left[\frac{1}{2}(u_t^2 + m^2u^2 + c^2|\nabla u|^2)\right]\right\} dV = \iiiint_W 0\, dV$$

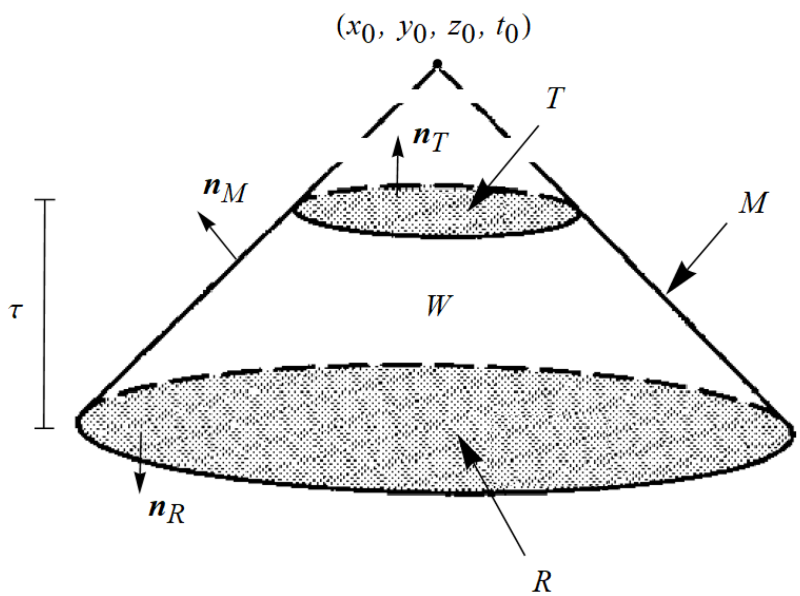

Figure 3: This is an illustration of the frustum $W = \{(x,y,z,t) \mid (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 \leq c^2(t_0-t)^2,\ 0 \leq t \leq \tau\}$. Its boundary consists of the bottom $R = \{(x,y,z) \mid (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 \leq c^2t_0^2\}$, the top $T = \{(x,y,z) \mid (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 \leq c^2(t_0-\tau)^2\}$, and the mantle $M = \{(x,y,z,t) \mid (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = c^2(t_0-t)^2,\ 0 \leq t \leq \tau\}$. τ represents the adjustable (temporal) height of the frustum: $0 < \tau < t_0$. In addition, $\mathbf{n}_R$, $\mathbf{n}_T$, and $\mathbf{n}_M$ are the outward unit vectors normal to the respective faces.

The integrand in curly braces is the divergence of a vector (treating t as if it were another spatial dimension).

$$\iiiint_W \nabla \cdot \left\langle -c^2 u_t u_x, -c^2 u_t u_y, -c^2 u_t u_z, \frac{1}{2}(u_t^2 + m^2u^2 + c^2|\nabla u|^2)\right\rangle dV = 0$$

Apply the divergence theorem to turn this volume integral into a surface integral over the frustum's boundary bdy W.

$$\oiiint_{\text{bdy } W} \left\langle -c^2 u_t u_x, -c^2 u_t u_y, -c^2 u_t u_z, \frac{1}{2}(u_t^2 + m^2 u^2 + c^2 |\nabla u|^2) \right\rangle \cdot \mathbf{n} \, dS = 0$$

$\mathbf{n}$ is the outward unit vector normal to the frustum. Split up the closed surface integral into a surface integral over each face of the frustum.

$$\begin{aligned} &\iiint_R \left\langle -c^2 u_t u_x, -c^2 u_t u_y, -c^2 u_t u_z, \frac{1}{2}(u_t^2 + m^2 u^2 + c^2 |\nabla u|^2) \right\rangle \cdot \mathbf{n}_R \, dS \\ &\quad + \iiint_T \left\langle -c^2 u_t u_x, -c^2 u_t u_y, -c^2 u_t u_z, \frac{1}{2}(u_t^2 + m^2 u^2 + c^2 |\nabla u|^2) \right\rangle \cdot \mathbf{n}_T \, dS \\ &\quad\quad + \iiint_M \left\langle -c^2 u_t u_x, -c^2 u_t u_y, -c^2 u_t u_z, \frac{1}{2}(u_t^2 + m^2 u^2 + c^2 |\nabla u|^2) \right\rangle \cdot \mathbf{n}_M \, dS = 0 \end{aligned} \tag{2}$$

Inspecting the frustum in Figure 3, the outward normal unit vectors are

$$\begin{aligned} \mathbf{n}_R &= \langle 0, 0, 0, -1 \rangle \\ \mathbf{n}_T &= \langle 0, 0, 0, 1 \rangle \\ \mathbf{n}_M &= \frac{\nabla \phi}{|\nabla \phi|}, \end{aligned}$$

where $\phi = \phi(x, y, z, t)$ is a level curve of the characteristic cone in equation (1).

$$\phi(x, y, z, t) = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 - c^2 (t - t_0)^2 = 0$$

As a result,

$$\begin{aligned}\mathbf{n}_M &= \frac{\langle 2(x-x_0), 2(y-y_0), 2(z-z_0), -2c^2(t-t_0)\rangle}{\sqrt{[2(x-x_0)]^2+[2(y-y_0)]^2+[2(z-z_0)]^2+[-2c^2(t-t_0)]^2}} \\ &= \frac{\langle x-x_0, y-y_0, z-z_0, c^2(t_0-t)\rangle}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2+c^4(t-t_0)^2}} \\ &= \left\langle \frac{x-x_0}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2+c^4(t-t_0)^2}}, \frac{y-y_0}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2+c^4(t-t_0)^2}}, \right. \\ &\qquad \left. \frac{z-z_0}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2+c^4(t-t_0)^2}}, \frac{c^2(t_0-t)}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2+c^4(t-t_0)^2}} \right\rangle.\end{aligned}$$

Use equation (1) to eliminate $c^2(t-t_0)^2$ in the first three components and to eliminate $(x-x_0)^2+(y-y_0)^2+(z-z_0)^2$ in the fourth component.

$$\begin{aligned} &= \left\langle \frac{x-x_0}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2+c^2[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]}}, \right. \\ &\qquad \frac{y-y_0}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2+c^2[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]}}, \\ &\qquad \left. \frac{z-z_0}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2+c^2[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]}}, \frac{c^2(t_0-t)}{\sqrt{c^2(t-t_0)^2+c^4(t-t_0)^2}} \right\rangle \\ &= \left\langle \frac{x-x_0}{\sqrt{(1+c^2)[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]}}, \frac{y-y_0}{\sqrt{(1+c^2)[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]}}, \right. \\ &\qquad \left. \frac{z-z_0}{\sqrt{(1+c^2)[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]}}, \frac{c^2(t_0-t)}{\sqrt{c^2(1+c^2)(t-t_0)^2}} \right\rangle \\ &= \frac{1}{\sqrt{1+c^2}} \left\langle \frac{x-x_0}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}, \frac{y-y_0}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}, \frac{z-z_0}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}, \frac{c(t_0-t)}{|t-t_0|} \right\rangle\end{aligned}$$

Within the frustum $t < t_0$, so $|t-t_0| = -(t-t_0) = t_0 - t$.

$$= \frac{c}{\sqrt{1+c^2}} \left\langle \frac{x-x_0}{c\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}, \frac{y-y_0}{c\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}, \frac{z-z_0}{c\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}, 1 \right\rangle$$

With these unit vectors equation (2) becomes

$$\begin{aligned}
&\iiint_R \left\langle -c^2u_tu_x, -c^2u_tu_y, -c^2u_tu_z, \frac{1}{2}(u_t^2 + m^2u^2 + c^2|\nabla u|^2)\right\rangle \cdot \langle 0,0,0,-1\rangle\, dS \\
&\quad + \iiint_T \left\langle -c^2u_tu_x, -c^2u_tu_y, -c^2u_tu_z, \frac{1}{2}(u_t^2 + m^2u^2 + c^2|\nabla u|^2)\right\rangle \cdot \langle 0,0,0,1\rangle\, dS \\
&\quad\quad + \iiint_M \left\langle -c^2u_tu_x, -c^2u_tu_y, -c^2u_tu_z, \frac{1}{2}(u_t^2 + m^2u^2 + c^2|\nabla u|^2)\right\rangle \\
&\quad\quad\quad \cdot \frac{c}{\sqrt{1+c^2}}\left\langle \frac{x-x_0}{c\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}, \frac{y-y_0}{c\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}},\right. \\
&\quad\quad\quad\quad \left. \frac{z-z_0}{c\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}, 1\right\rangle dS = 0
\end{aligned}$$

$$\begin{aligned}
&\iiint_R \frac{1}{2}(u_t^2 + m^2u^2 + c^2|\nabla u|^2)(-1)\, dS \\
&\quad + \iiint_T \frac{1}{2}(u_t^2 + m^2u^2 + c^2|\nabla u|^2)\, dS \\
&\quad\quad + \frac{c}{\sqrt{1+c^2}}\iiint_M \left[(-c^2u_tu_x)\frac{x-x_0}{c\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}} + (-c^2u_tu_y)\frac{y-y_0}{c\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}\right. \\
&\quad\quad\quad \left. +(-c^2u_tu_z)\frac{z-z_0}{c\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}} + \frac{1}{2}(u_t^2 + m^2u^2 + c^2|\nabla u|^2)\right] dS = 0
\end{aligned}$$

$$\begin{aligned}
&-\iiint_R \frac{1}{2}(u_t^2 + m^2u^2 + c^2|\nabla u|^2)\, dS + \iiint_T \frac{1}{2}(u_t^2 + m^2u^2 + c^2|\nabla u|^2)\, dS \\
&\quad + \frac{c}{\sqrt{1+c^2}}\iiint_M \left\{(-cu_t)\left[u_x\frac{x-x_0}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}} + u_y\frac{y-y_0}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}\right.\right. \\
&\quad\quad \left.\left. +u_z\frac{z-z_0}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}\right] + \frac{1}{2}(u_t^2 + m^2u^2 + c^2|\nabla u|^2)\right\} dS = 0. \qquad (3)
\end{aligned}$$

The integral over the mantle M will now be shown to be positive (or zero).

$$\frac{c}{\sqrt{1+c^2}}\iiint_M \left[(-cu_t)\langle u_x, u_y, u_z\rangle \cdot \frac{\langle x-x_0, y-y_0, z-z_0\rangle}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}} + \frac{1}{2}(u_t^2+m^2u^2+c^2|\nabla u|^2)\right] dS$$

If we let $\boldsymbol{\imath} = \langle x-x_0, y-y_0, z-z_0\rangle$, then the dot product can be interpreted as a directional derivative in the direction of $\hat{\boldsymbol{\imath}}$: $\nabla u \cdot \hat{\boldsymbol{\imath}} = \partial u/\partial \imath$.

$$\begin{aligned}
&= \frac{c}{\sqrt{1+c^2}}\iiint_M \left[(-cu_t)\nabla u \cdot \hat{\boldsymbol{\imath}} + \frac{1}{2}(u_t^2+m^2u^2+c^2|\nabla u|^2)\right] dS \\
&= \frac{c}{\sqrt{1+c^2}}\iiint_M \left[-cu_t u_{\imath} + \frac{1}{2}(u_t^2+m^2u^2+c^2|\nabla u|^2)\right] dS \\
&= \frac{c}{\sqrt{1+c^2}}\iiint_M \left[\frac{1}{2}(u_t^2 - 2cu_t u_{\imath}) + \frac{1}{2}m^2u^2 + \frac{1}{2}c^2|\nabla u|^2\right] dS
\end{aligned}$$

Complete the square.

$$\begin{aligned}
&= \frac{c}{\sqrt{1+c^2}}\iiint_M \left[\frac{1}{2}(u_t^2 - 2cu_t u_{\imath} + c^2u_{\imath}^2) + \frac{1}{2}m^2u^2 + \frac{1}{2}c^2|\nabla u|^2 - \frac{1}{2}c^2u_{\imath}^2\right] dS \\
&= \frac{c}{\sqrt{1+c^2}}\iiint_M \left[\frac{1}{2}(u_t - cu_{\imath})^2 + \frac{1}{2}m^2u^2 + \frac{1}{2}c^2(|\nabla u|^2 - u_{\imath}^2)\right] dS \\
&= \frac{c}{\sqrt{1+c^2}}\iiint_M \left[\frac{1}{2}(u_t - cu_{\imath})^2 + \frac{1}{2}m^2u^2 + \frac{1}{2}c^2(\nabla u - u_{\imath}\hat{\boldsymbol{\imath}})\cdot(\nabla u - u_{\imath}\hat{\boldsymbol{\imath}})\right] dS \\
&= \frac{c}{\sqrt{1+c^2}}\iiint_M \left[\frac{1}{2}(u_t - cu_{\imath})^2 + \frac{1}{2}m^2u^2 + \frac{1}{2}c^2|\nabla u - u_{\imath}\hat{\boldsymbol{\imath}}|^2\right] dS
\end{aligned}$$

The integrand is a sum of squared terms, so the integral over M is positive (it could be equal to zero if u is zero, for example, if $\phi = \psi = 0$). Bringing the integral over R to the right side, equation (3) becomes

$$\begin{aligned}
\iiint_T \frac{1}{2}(u_t^2+m^2u^2+c^2|\nabla u|^2)\,dS + \frac{c}{\sqrt{1+c^2}}\iiint_M \left[\frac{1}{2}(u_t - cu_{\imath})^2 + \frac{1}{2}m^2u^2 + \frac{1}{2}c^2|\nabla u - u_{\imath}\hat{\boldsymbol{\imath}}|^2\right] dS \\
= \iiint_R \frac{1}{2}(u_t^2+m^2u^2+c^2|\nabla u|^2)\,dS
\end{aligned}$$

from which we conclude that

$$\iiint_T \frac{1}{2}(u_t^2+m^2u^2+c^2|\nabla u|^2)\,dS \le \iiint_R \frac{1}{2}(u_t^2+m^2u^2+c^2|\nabla u|^2)\,dS.$$

The R face of the frustum lies in the $t=0$ plane, so u and u_t can be replaced by ϕ and ψ, respectively.

$$\iiint_T \frac{1}{2}(u_t^2+m^2u^2+c^2|\nabla u|^2)\,dS \le \iiint_R \frac{1}{2}(\psi^2+m^2\phi^2+c^2|\nabla \phi|^2)\,dS$$

Substitute the formulas for T and R and note that dS is $dx\,dy\,dz$ on the top and bottom faces of the frustum.

$$\iiint\limits_{\substack{(x-x_0)^2+(y-y_0)^2\\+(z-z_0)^2\leq c^2(t_0-\tau)^2}} \frac{1}{2}(u_t^2+m^2u^2+c^2|\nabla u|^2)\,dx\,dy\,dz \leq \iiint\limits_{\substack{(x-x_0)^2+(y-y_0)^2\\+(z-z_0)^2\leq c^2t_0^2}} \frac{1}{2}(\psi^2+m^2\phi^2+c^2|\nabla\phi|^2)\,dx\,dy\,dz$$

These triple integrals can be written explicitly by using spherical coordinates $(\mathscr{r},\phi,\theta)$, where θ represents the angle from the polar axis.

$$\begin{aligned} x-x_0 &= \mathscr{r}\sin\theta\cos\phi \\ y-y_0 &= \mathscr{r}\sin\theta\sin\phi \\ z-z_0 &= \mathscr{r}\cos\theta \end{aligned}$$

Therefore, the principle of causality for the Klein–Gordon equation in three dimensions is

$$\int_0^\pi\int_0^{2\pi}\int_0^{c(t_0-\tau)} \frac{1}{2}(u_t^2+m^2u^2+c^2|\nabla u|^2)\mathscr{r}^2\sin\theta\,d\mathscr{r}\,d\phi\,d\theta \leq \int_0^\pi\int_0^{2\pi}\int_0^{ct_0} \frac{1}{2}(\psi^2+m^2\phi^2+c^2|\nabla\phi|^2)\mathscr{r}^2\sin\theta\,d\mathscr{r}\,d\phi\,d\theta,$$

where $0<\tau<t_0$. The energy within a solid ball of radius $c(t_0-t)$ is

$$\mathscr{E}(t) = \int_0^\pi\int_0^{2\pi}\int_0^{c(t_0-t)} \frac{1}{2}(u_t^2+m^2u^2+c^2|\nabla u|^2)\mathscr{r}^2\sin\theta\,d\mathscr{r}\,d\phi\,d\theta,$$

so the causality principle can be expressed compactly as

$$\mathscr{E}(\tau)\leq\mathscr{E}(0).$$

This implies that the energy at a particular point (x_0,y_0,z_0,t_0) cannot be larger than what is initially within the hyperdisk $(x-x_0)^2+(y-y_0)^2+(z-z_0)^2\leq c^2t_0^2$. It follows that c is the maximum possible speed a solution of the Klein–Gordon equation can travel because otherwise points in the xyz-plane outside the hyperdisk would also contribute to the energy at (x_0,y_0,z_0,t_0).