Exercise 10

Derive the mean value property of harmonic functions $u(x, y, z)$ by the following method. A harmonic function is a wave that happens not to depend on time, so that its mean value $\overline{u}(r, t) = \overline{u}(r)$ satisfies (5). Deduce that $\overline{u}(r) = u(\mathbf{0})$.

Solution

$\overline{u}(r, t)$ is defined to be the average of u over a spherical surface of radius r. Spherical coordinates (r, ϕ, θ) will be used to write the forthcoming integrals explicitly, where θ represents the angle from the polar axis.

$$\overline{u}(r,t) = \frac{\iint u\,dS}{\iint dS} = \frac{\int_0^\pi \int_0^{2\pi} u(r,\phi,\theta,t)(r^2 \sin\theta\,d\phi\,d\theta)}{\int_0^\pi \int_0^{2\pi} (r^2 \sin\theta\,d\phi\,d\theta)} = \frac{\cancel{r^2} \int_0^\pi \int_0^{2\pi} u(r,\phi,\theta,t) \sin\theta\,d\phi\,d\theta}{\cancel{r^2} \left(\int_0^\pi \sin\theta\,d\theta\right) \left(\int_0^{2\pi} d\phi\right)}$$

Evaluating the integrals in the denominator, we have

$$\overline{u}(r,t) = \frac{1}{4\pi} \int_0^\pi \int_0^{2\pi} u(r,\phi,\theta,t) \sin\theta\,d\phi\,d\theta. \tag{1}$$

The three-dimensional wave equation in space is

$$u_{tt} = c^2 \nabla^2 u, \quad -\infty < x, y, z < \infty,\ t > 0.$$

Integrate both sides over the volume enclosed by that sphere of radius r.

$$\iiint_V u_{tt}\,dV = \iiint_V c^2 \nabla^2 u\,dV$$

$$\iiint_V u_{tt}\,dV = c^2 \iiint_V \nabla \cdot \nabla u\,dV$$

Apply the divergence theorem to the volume integral on the right side to turn it into a surface integral over the sphere's boundary.

$$\iiint_V u_{tt}\,dV = c^2 \oiint_S \nabla u \cdot \hat{\mathbf{n}}\,dS$$

The unit vector normal to the boundary is the radial unit vector: $\hat{\mathbf{n}} = \hat{\mathbf{r}}$. $\nabla u \cdot \hat{\mathbf{r}}$ can be interpreted as the directional derivative in the radial direction, that is, $\partial u/\partial r$.

$$\iiint_V \frac{\partial^2 u}{\partial t^2}\,dV = c^2 \oiint_S \frac{\partial u}{\partial r}\,dS$$

$$\int_0^\pi \int_0^{2\pi} \int_0^r \frac{\partial^2 u}{\partial t^2}(\rho^2 \sin\theta\,d\rho\,d\phi\,d\theta) = c^2 \int_0^\pi \int_0^{2\pi} \frac{\partial u}{\partial r}(r^2 \sin\theta\,d\phi\,d\theta)$$

$$\int_0^r \rho^2 \int_0^\pi \int_0^{2\pi} \frac{\partial^2 u}{\partial t^2} \sin\theta\,d\phi\,d\theta\,d\rho = c^2 r^2 \int_0^\pi \int_0^{2\pi} \frac{\partial u}{\partial r} \sin\theta\,d\phi\,d\theta$$

$$\int_0^r \rho^2 \frac{\partial^2}{\partial t^2}\left[\int_0^\pi \int_0^{2\pi} u(\rho,\phi,\theta,t)\sin\theta\,d\phi\,d\theta\right]d\rho = c^2 r^2 \frac{\partial}{\partial r}\left[\int_0^\pi \int_0^{2\pi} u(r,\phi,\theta,t)\sin\theta\,d\phi\,d\theta\right]$$

Divide both sides by 4π.

$$\int_0^r \rho^2 \frac{\partial^2}{\partial t^2}\left[\frac{1}{4\pi}\int_0^\pi \int_0^{2\pi} u(\rho,\phi,\theta,t)\sin\theta\,d\phi\,d\theta\right]d\rho = c^2 r^2 \frac{\partial}{\partial r}\left[\frac{1}{4\pi}\int_0^\pi \int_0^{2\pi} u(r,\phi,\theta,t)\sin\theta\,d\phi\,d\theta\right]$$

Substitute equation (1) here.

$$\int_0^r \rho^2 \frac{\partial^2 \overline{u}}{\partial t^2}\,d\rho = c^2 r^2 \frac{\partial \overline{u}}{\partial r}$$

Suppose now that u is a harmonic function, a wave that does not depend on time. Then $\overline{u} = \overline{u}(r)$, and the left side is equal to zero.

$$0 = c^2 r^2 \frac{d\overline{u}}{dr}$$

Divide both sides by $c^2 r^2$.

$$\frac{d\overline{u}}{dr} = 0$$

Integrate both sides with respect to r.

$$\overline{u}(r) = C_1$$

We conclude that the average of u (a harmonic function) over a spherical surface is the same at every radius, including $r = 0$.

$$\begin{aligned}\overline{u}(r) &= \overline{u}(0)\\ &= \frac{1}{4\pi}\int_0^\pi \int_0^{2\pi} u(0,\phi,\theta)\sin\theta\,d\phi\,d\theta\end{aligned}$$

Note that u evaluated at $r = 0$ is $u(x = 0, y = 0, z = 0)$, or $u(\mathbf{0})$, and does not depend on ϕ or θ.

$$\begin{aligned}&= \frac{1}{4\pi}\int_0^\pi \int_0^{2\pi} u(\mathbf{0})\sin\theta\,d\phi\,d\theta\\ &= \frac{u(\mathbf{0})}{4\pi}\int_0^\pi \int_0^{2\pi} \sin\theta\,d\phi\,d\theta\\ &= \frac{u(\mathbf{0})}{4\pi}\left(\int_0^\pi \sin\theta\,d\theta\right)\left(\int_0^{2\pi} d\phi\right)\\ &= \frac{u(\mathbf{0})}{4\pi}(2)(2\pi)\end{aligned}$$

Therefore,

$$\overline{u}(r) = u(\mathbf{0}).$$