Exercise 11

Find all the spherical solutions of the three-dimensional wave equation; that is, find the solutions that depend only on r and t. (*Hint:* See (5).)

Solution

The three-dimensional wave equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u.$$

Expand the Laplacian operator in spherical coordinates (r, ϕ, θ). Here θ represents the angle from the polar axis.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left\{ \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\left[\frac{\partial^2 u}{\partial \theta^2} + (\cot\theta)\frac{\partial u}{\partial \theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2}\right]\right\}$$

Assuming the solution is spherically symmetric, u is only a function of r and t. As a result, the angular derivatives vanish.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}\right)$$

Multiply both sides by r.

$$r\frac{\partial^2 u}{\partial t^2} = c^2 \left(r\frac{\partial^2 u}{\partial r^2} + 2\frac{\partial u}{\partial r}\right)$$

Now make the change of variables $w(r,t) = ru(r,t)$. Write the new derivatives in terms of the old ones.

$$\begin{aligned}
\frac{\partial^2 w}{\partial t^2} &= r\frac{\partial^2 u}{\partial t^2} \\
\frac{\partial w}{\partial r} &= u + r\frac{\partial u}{\partial r} \\
\frac{\partial^2 w}{\partial r^2} &= 2\frac{\partial u}{\partial r} + r\frac{\partial^2 u}{\partial r^2}
\end{aligned}$$

The transformed PDE is the one-dimensional wave equation.

$$\frac{\partial^2 w}{\partial t^2} = c^2\frac{\partial^2 w}{\partial r^2}$$

The general solution is a sum of two waves, one travelling to the left at speed c and one travelling to the right at speed c.

$$w(r,t) = f(r+ct) + g(r-ct)$$

f and g are arbitrary functions. Therefore, since $u = w/r$,

$$u(r,t) = \frac{f(r+ct) + g(r-ct)}{r}.$$

Note that u falls off as $1/r$ in three dimensions.