Exercise 17

Use the result of Exercise 16 to compute the limit of $t \cdot u(\mathbf{0}, t)$ as $t \to \infty$.

Solution

From Exercise 16, the solution for $u(0, 0, t)$ was found to be

$$u(0,0,t) = \begin{cases} At & \text{if } t \leq \frac{\rho}{c} \\ A\left(t - \sqrt{t^2 - \frac{\rho^2}{c^2}}\right) & \text{if } t \geq \frac{\rho}{c} \end{cases}.$$

Since we are taking the limit as $t \to \infty$, the olive solution applies.

$$\begin{aligned} \lim_{t\to\infty} t \cdot u(\mathbf{0}, t) &= \lim_{t\to\infty} t \cdot A\left(t - \sqrt{t^2 - \frac{\rho^2}{c^2}}\right) \\ &= \lim_{t\to\infty} At\left(t - t\sqrt{1 - \frac{\rho^2}{c^2t^2}}\right) \\ &= \lim_{t\to\infty} At^2\left(1 - \sqrt{1 - \frac{\rho^2}{c^2t^2}}\right) \end{aligned}$$

Use the binomial series for the square root.

$$\begin{aligned} &= \lim_{t\to\infty} At^2\left\{1 - \left[1 - \frac{\rho^2}{2c^2t^2} + O\left(\frac{1}{t^4}\right)\right]\right\} \\ &= \lim_{t\to\infty} At^2\left[\frac{\rho^2}{2c^2t^2} - O\left(\frac{1}{t^4}\right)\right] \\ &= \lim_{t\to\infty} \left[\frac{A\rho^2}{2c^2} - \cancelto{0}{O\left(\frac{1}{t^2}\right)}\right] \end{aligned}$$

Therefore,

$$\lim_{t\to\infty} t \cdot u(\mathbf{0}, t) = \frac{A\rho^2}{2c^2}.$$