Exercise 20

"Descend" from two dimensions to one as follows. Let $u_{tt} = c^2 u_{xx}$ with initial data $\phi(x) \equiv 0$ and general $\psi(x)$. Imagine that we don't know d'Alembert's solution formula. Think of $u(x,t)$ as a solution of the two-dimensional equation that happens not to depend on y. Plug it into (19) and carry out the integration.

Solution

Equation (19) in the textbook gives the solution to the two-dimensional wave equation with the initial data, $u = 0$ and $u_t = \psi$, at a particular point (x_0, y_0, t_0).

$$u(x_0, y_0, t_0) = \frac{1}{2\pi c} \iint\limits_{\substack{(x-x_0)^2+(y-y_0)^2 \\ \leq c^2 t_0^2}} \frac{\psi(x,y)}{\left[c^2 t_0^2 - (x-x_0)^2 - (y-y_0)^2\right]^{1/2}}\, dx\, dy \tag{19}$$

This double integral is over a disk in the xy-plane centered at (x_0, y_0) with radius ct_0. Suppose that u and ψ do not depend on y.

$$u(x_0, t_0) = \frac{1}{2\pi c} \iint\limits_{\substack{(x-x_0)^2+(y-y_0)^2 \\ \leq c^2 t_0^2}} \frac{\psi(x)}{\sqrt{c^2 t_0^2 - (x-x_0)^2 - (y-y_0)^2}}\, dx\, dy$$

The integral in dy is now one we can evaluate.

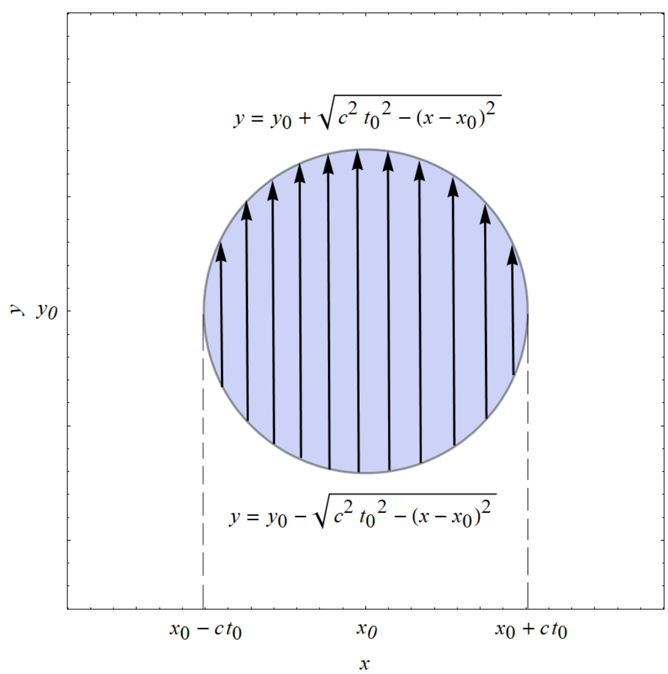

Figure 1: The shaded region is the disk $(x-x_0)^2 + (y-y_0)^2 \leq c^2 t_0^2$ centered at (x_0, y_0) with radius ct_0. The lower half of it is represented by $y = y_0 - \sqrt{c^2 t_0^2 - (x-x_0)^2}$, and the upper half of it is represented by $y = y_0 + \sqrt{c^2 t_0^2 - (x-x_0)^2}$.

Integrate over the disk in the xy-plane as shown in Figure 1 so that dy comes first.

$$\begin{aligned}
u(x_0,t_0) &= \frac{1}{2\pi c}\int_{x_0-ct_0}^{x_0+ct_0}\int_{y_0-\sqrt{c^2t_0^2-(x-x_0)^2}}^{y_0+\sqrt{c^2t_0^2-(x-x_0)^2}} \frac{\psi(x)}{\sqrt{c^2t_0^2-(x-x_0)^2-(y-y_0)^2}}\,dy\,dx \\
&= \frac{1}{2\pi c}\int_{x_0-ct_0}^{x_0+ct_0}\psi(x)\left[\int_{y_0-\sqrt{c^2t_0^2-(x-x_0)^2}}^{y_0+\sqrt{c^2t_0^2-(x-x_0)^2}} \frac{dy}{\sqrt{c^2t_0^2-(x-x_0)^2-(y-y_0)^2}}\right]dx
\end{aligned}$$

Let $m = y - y_0$ in the integral.

$$= \frac{1}{2\pi c}\int_{x_0-ct_0}^{x_0+ct_0}\psi(x)\left[\int_{-\sqrt{c^2t_0^2-(x-x_0)^2}}^{\sqrt{c^2t_0^2-(x-x_0)^2}} \frac{dm}{\sqrt{c^2t_0^2-(x-x_0)^2-m^2}}\right]dx$$

Let $p^2 = c^2t_0^2 - (x-x_0)^2$.

$$= \frac{1}{2\pi c}\int_{x_0-ct_0}^{x_0+ct_0}\psi(x)\left(\int_{-p}^{p}\frac{dm}{\sqrt{p^2-m^2}}\right)dx$$

Use trigonometric substitution to evaluate the integral in parentheses.

$$\begin{aligned}
m &= p\cos\theta \quad \rightarrow \quad p^2-m^2 = p^2-p^2\cos^2\theta = p^2\sin^2\theta \quad \rightarrow \quad \sqrt{p^2-m^2} = p\sin\theta \\
dm &= -p\sin\theta\,d\theta
\end{aligned}$$

The integral becomes

$$\begin{aligned}
u(x_0,t_0) &= \frac{1}{2\pi c}\int_{x_0-ct_0}^{x_0+ct_0}\psi(x)\left(\int_{\cos^{-1}(-1)}^{\cos^{-1}(1)}\frac{-p\sin\theta\,d\theta}{p\sin\theta}\right)dx \\
&= \frac{1}{2\pi c}\int_{x_0-ct_0}^{x_0+ct_0}\psi(x)\left(-\int_{\pi}^{0}d\theta\right)dx \\
&= \frac{1}{2\pi c}\int_{x_0-ct_0}^{x_0+ct_0}\psi(x)(\pi)\,dx \\
&= \frac{1}{2c}\int_{x_0-ct_0}^{x_0+ct_0}\psi(x)\,dx.
\end{aligned}$$

Therefore, switching the roles of x and x_0, we obtain the solution predicted by d'Alembert's formula.

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(x_0)\,dx_0$$

General Initial Conditions

The aim here is to obtain the solution to the one-dimensional wave equation subject to two initial conditions by descending from the second dimension. The solution to the two-dimensional wave equation with two initial conditions,

$$\begin{aligned} &u_{tt} = c^2\nabla^2 u, \quad -\infty < x, y < \infty,\ t > 0 \\ &u(x,y,0) = \phi(x,y) \\ &u_t(x,y,0) = \psi(x,y), \end{aligned}$$

is

$$\begin{aligned} u(x,y,t) = \frac{\partial}{\partial t}\left[\frac{1}{2\pi c}\iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\ \leq c^2t^2}} \frac{\phi(x_0,y_0)}{\sqrt{c^2t^2-(x_0-x)^2-(y_0-y)^2}}\,dx_0\,dy_0\right] \\ + \frac{1}{2\pi c}\iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\ \leq c^2t^2}} \frac{\psi(x_0,y_0)}{\sqrt{c^2t^2-(x_0-x)^2-(y_0-y)^2}}\,dx_0\,dy_0. \end{aligned}$$

These double integrals are over a disk in the x_0y_0-plane centered at (x, y) with radius ct. Suppose that u and ϕ and ψ do not depend on y.

$$\begin{aligned} u(x,t) = \frac{\partial}{\partial t}\left[\frac{1}{2\pi c}\iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\ \leq c^2t^2}} \frac{\phi(x_0)}{\sqrt{c^2t^2-(x_0-x)^2-(y_0-y)^2}}\,dx_0\,dy_0\right] \\ + \frac{1}{2\pi c}\iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\ \leq c^2t^2}} \frac{\psi(x_0)}{\sqrt{c^2t^2-(x_0-x)^2-(y_0-y)^2}}\,dx_0\,dy_0 \end{aligned}$$

The integrals in dy_0 can now be evaluated. Integrate over the disk in the x_0y_0-plane as shown in Figure 2 so that dy_0 comes first.

$$\begin{aligned} u(x,t) &= \frac{\partial}{\partial t}\left[\frac{1}{2\pi c}\int_{x-ct}^{x+ct}\int_{y-\sqrt{c^2t^2-(x_0-x)^2}}^{y+\sqrt{c^2t^2-(x_0-x)^2}} \frac{\phi(x_0)}{\sqrt{c^2t^2-(x_0-x)^2-(y_0-y)^2}}\,dy_0\,dx_0\right] \\ &\quad + \frac{1}{2\pi c}\int_{x-ct}^{x+ct}\int_{y-\sqrt{c^2t^2-(x_0-x)^2}}^{y+\sqrt{c^2t^2-(x_0-x)^2}} \frac{\psi(x_0)}{\sqrt{c^2t^2-(x_0-x)^2-(y_0-y)^2}}\,dy_0\,dx_0 \\ &= \frac{\partial}{\partial t}\left\{\frac{1}{2\pi c}\int_{x-ct}^{x+ct}\phi(x_0)\left[\int_{y-\sqrt{c^2t^2-(x_0-x)^2}}^{y+\sqrt{c^2t^2-(x_0-x)^2}} \frac{dy_0}{\sqrt{c^2t^2-(x_0-x)^2-(y_0-y)^2}}\right]dx_0\right\} \\ &\quad + \frac{1}{2\pi c}\int_{x-ct}^{x+ct}\psi(x_0)\left[\int_{y-\sqrt{c^2t^2-(x_0-x)^2}}^{y+\sqrt{c^2t^2-(x_0-x)^2}} \frac{dy_0}{\sqrt{c^2t^2-(x_0-x)^2-(y_0-y)^2}}\right]dx_0 \end{aligned}$$

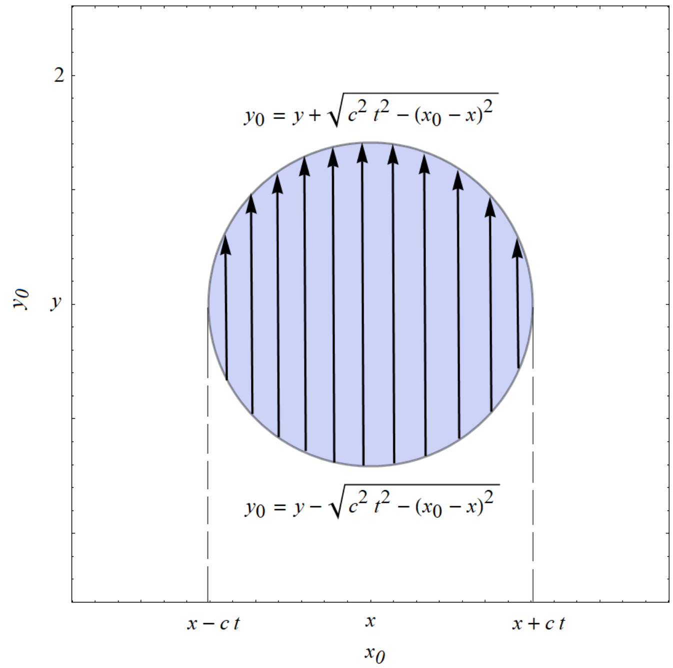

Figure 2: The shaded region is the disk $(x_0 - x)^2 + (y_0 - y)^2 \leq c^2t^2$ centered at (x_0, y_0) with radius ct_0. The lower half of it is represented by $y_0 = y - \sqrt{c^2t^2 - (x_0 - x)^2}$, and the upper half of it is represented by $y_0 = y + \sqrt{c^2t^2 - (x_0 - x)^2}$.

Let $n = y_0 - y$.

$$u(x,t) = \frac{\partial}{\partial t}\left\{\frac{1}{2\pi c}\int_{x-ct}^{x+ct}\phi(x_0)\left[\int_{-\sqrt{c^2t^2-(x_0-x)^2}}^{\sqrt{c^2t^2-(x_0-x)^2}}\frac{dn}{\sqrt{c^2t^2-(x_0-x)^2-n^2}}\right]dx_0\right\}$$
$$+\frac{1}{2\pi c}\int_{x-ct}^{x+ct}\psi(x_0)\left[\int_{-\sqrt{c^2t^2-(x_0-x)^2}}^{\sqrt{c^2t^2-(x_0-x)^2}}\frac{dn}{\sqrt{c^2t^2-(x_0-x)^2-n^2}}\right]dx_0$$

Let $h^2 = c^2t^2 - (x_0 - x)^2$.

$$= \frac{\partial}{\partial t}\left[\frac{1}{2\pi c}\int_{x-ct}^{x+ct}\phi(x_0)\left(\int_{-h}^{h}\frac{dn}{\sqrt{h^2-n^2}}\right)dx_0\right] + \frac{1}{2\pi c}\int_{x-ct}^{x+ct}\psi(x_0)\left(\int_{-h}^{h}\frac{dn}{\sqrt{h^2-n^2}}\right)dx_0$$
$$= \frac{\partial}{\partial t}\left[\frac{1}{2\pi c}\int_{x-ct}^{x+ct}\phi(x_0)(\pi)\,dx_0\right] + \frac{1}{2\pi c}\int_{x-ct}^{x+ct}\psi(x_0)(\pi)\,dx_0$$
$$= \frac{\partial}{\partial t}\left[\frac{1}{2c}\int_{x-ct}^{x+ct}\phi(x_0)\,dx_0\right] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(x_0)\,dx_0$$
$$= \frac{1}{2c}\frac{\partial}{\partial t}\int_{x-ct}^{x+ct}\phi(x_0)\,dx_0 + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(x_0)\,dx_0$$

Apply the Leibnitz rule to differentiate the integral and obtain d'Alembert's formula.

$$= \frac{1}{2c}[\phi(x+ct)(c) - \phi(x-ct)(-c)] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(x_0)\,dx_0$$
$$= \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(x_0)\,dx_0$$