

Exercise 3

Solve the wave equation in three dimensions with the initial data $\phi \equiv 0$, $\psi(x, y, z) = y$, by use of (3).

Solution

Equation (3) is the solution to the three-dimensional wave equation in space subject to two initial conditions.

$$\begin{aligned}u_{tt} &= c^2 \nabla^2 u, & -\infty < x, y, z < \infty, & t > 0 \\u(x, y, z, 0) &= \alpha(x, y, z) \\u_t(x, y, z, 0) &= \beta(x, y, z)\end{aligned}$$

Integrate both sides of the wave equation over the black hyperdisk shown in the figure below.

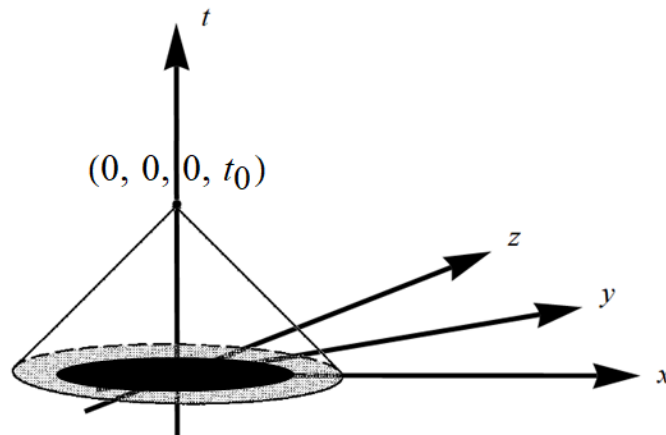


Figure 1: The black hyperdisk lies in the xyz -plane, has center $(0, 0, 0)$ and radius r , and represents a solid ball in xyz -space. Note that the shaded hyperdisk it lies within has radius ct_0 , where t_0 is a particular time we want to evaluate u at.

$$\begin{aligned}\iiint_V u_{tt} dV &= \iiint_V c^2 \nabla^2 u dV \\ \iiint_V u_{tt} dV &= c^2 \iiint_V \nabla \cdot \nabla u dV\end{aligned}$$

Apply the divergence theorem to the volume integral on the right side to turn it into a surface integral over the solid ball's boundary.

$$\iiint_V u_{tt} dV = c^2 \oint_S \nabla u \cdot \hat{\mathbf{n}} dS$$

The unit vector normal to the boundary is the radial unit vector: $\hat{\mathbf{n}} = \hat{\mathbf{r}}$. $\nabla u \cdot \hat{\mathbf{r}}$ can be interpreted as the directional derivative in the radial direction, that is, $\partial u / \partial r$.

$$\iiint_V \frac{\partial^2 u}{\partial t^2} dV = c^2 \oint_S \frac{\partial u}{\partial r} dS$$

Write out the volume and surface integrals explicitly by using spherical coordinates (r, ϕ, θ) . Here θ denotes the angle from the polar axis.

$$\begin{aligned} \int_0^\pi \int_0^{2\pi} \int_0^r \frac{\partial^2 u}{\partial t^2} (\rho^2 \sin \theta \, d\rho \, d\phi \, d\theta) &= c^2 \int_0^\pi \int_0^{2\pi} \frac{\partial u}{\partial r} (r^2 \sin \theta \, d\phi \, d\theta) \\ \int_0^r \rho^2 \int_0^\pi \int_0^{2\pi} \frac{\partial^2 u}{\partial t^2} \sin \theta \, d\phi \, d\theta \, d\rho &= c^2 r^2 \int_0^\pi \int_0^{2\pi} \frac{\partial u}{\partial r} \sin \theta \, d\phi \, d\theta \\ \int_0^r \rho^2 \frac{\partial^2}{\partial t^2} \left[\int_0^\pi \int_0^{2\pi} u(\rho, \phi, \theta, t) \sin \theta \, d\phi \, d\theta \right] d\rho &= c^2 r^2 \frac{\partial}{\partial r} \left[\int_0^\pi \int_0^{2\pi} u(r, \phi, \theta, t) \sin \theta \, d\phi \, d\theta \right] \end{aligned}$$

Let

$$v(r, t) = \int_0^\pi \int_0^{2\pi} u(r, \phi, \theta, t) \sin \theta \, d\phi \, d\theta$$

so that the previous equation becomes

$$\int_0^r \rho^2 \frac{\partial^2 v}{\partial t^2} d\rho = c^2 r^2 \frac{\partial v}{\partial r}.$$

Differentiate both sides with respect to r to eliminate the integral on the left side.

$$r^2 \frac{\partial^2 v}{\partial t^2} = c^2 \left(2r \frac{\partial v}{\partial r} + r^2 \frac{\partial^2 v}{\partial r^2} \right)$$

Divide both sides by r .

$$r \frac{\partial^2 v}{\partial t^2} = c^2 \left(2 \frac{\partial v}{\partial r} + r \frac{\partial^2 v}{\partial r^2} \right)$$

Now make the change of variables $w(r, t) = rv(r, t)$. Write the new derivatives in terms of the old ones by differentiating this substitution.

$$\begin{aligned} \frac{\partial^2 w}{\partial t^2} &= r \frac{\partial^2 v}{\partial t^2} \\ \frac{\partial w}{\partial r} &= v + r \frac{\partial v}{\partial r} \\ \frac{\partial^2 w}{\partial r^2} &= 2 \frac{\partial v}{\partial r} + r \frac{\partial^2 v}{\partial r^2} \end{aligned}$$

The transformed PDE is the wave equation on a semi-infinite interval

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial r^2}, \quad 0 < r < \infty, \quad t > 0$$

subject to the Dirichlet boundary condition $w(0, t) = 0$ and the initial conditions,

$$\begin{aligned} w(r, 0) &= r \int_0^\pi \int_0^{2\pi} u(r, \phi, \theta, 0) \sin \theta \, d\phi \, d\theta & \text{and} & & w_t(r, 0) &= r \int_0^\pi \int_0^{2\pi} u_t(r, \phi, \theta, 0) \sin \theta \, d\phi \, d\theta \\ &= r \int_0^\pi \int_0^{2\pi} \alpha(r, \phi, \theta) \sin \theta \, d\phi \, d\theta & & & &= r \int_0^\pi \int_0^{2\pi} \beta(r, \phi, \theta) \sin \theta \, d\phi \, d\theta. \end{aligned}$$

The method of reflection can be applied to solve this one-dimensional wave equation. Consider the corresponding problem over the whole line,

$$W_{tt} = c^2 W_{rr}, \quad -\infty < r < \infty, \quad t > 0$$

$$W(r, 0) = A_{\text{odd}}(r), \quad W_t(r, 0) = B_{\text{odd}}(r),$$

where the odd extensions of the initial conditions for w , $A_{\text{odd}}(r)$ and $B_{\text{odd}}(r)$, are used in order to satisfy the Dirichlet boundary condition.

$$A_{\text{odd}}(r) = \begin{cases} r \int_0^\pi \int_0^{2\pi} \alpha(r, \phi, \theta) \sin \theta \, d\phi \, d\theta & \text{if } r > 0 \\ -(-r) \int_0^\pi \int_0^{2\pi} \alpha(-r, \phi, \theta) \sin \theta \, d\phi \, d\theta & \text{if } r < 0 \end{cases}$$

$$B_{\text{odd}}(r) = \begin{cases} r \int_0^\pi \int_0^{2\pi} \beta(r, \phi, \theta) \sin \theta \, d\phi \, d\theta & \text{if } r > 0 \\ -(-r) \int_0^\pi \int_0^{2\pi} \beta(-r, \phi, \theta) \sin \theta \, d\phi \, d\theta & \text{if } r < 0 \end{cases}$$

The solution for W is given by d'Alembert's formula in section 2.1 on page 36.

$$W(r, t) = \frac{1}{2}[A_{\text{odd}}(r + ct) + A_{\text{odd}}(r - ct)] + \frac{1}{2c} \int_{r-ct}^{r+ct} B_{\text{odd}}(s) \, ds$$

The solution for w is then just the restriction of W to $r > 0$.

$$w(r, t) = \frac{1}{2}[A_{\text{odd}}(r + ct) + A_{\text{odd}}(r - ct)] + \frac{1}{2c} \int_{r-ct}^{r+ct} B_{\text{odd}}(s) \, ds, \quad r > 0$$

Our task now is to write this formula in terms of the given functions, α and β . Note that

$$A_{\text{odd}}(r + ct) = \begin{cases} (r + ct) \int_0^\pi \int_0^{2\pi} \alpha(r + ct, \phi, \theta) \sin \theta \, d\phi \, d\theta & \text{if } r + ct > 0 \\ -(-r - ct) \int_0^\pi \int_0^{2\pi} \alpha(-r - ct, \phi, \theta) \sin \theta \, d\phi \, d\theta & \text{if } r + ct < 0 \end{cases}$$

and

$$A_{\text{odd}}(r - ct) = \begin{cases} (r - ct) \int_0^\pi \int_0^{2\pi} \alpha(r - ct, \phi, \theta) \sin \theta \, d\phi \, d\theta & \text{if } r - ct > 0 \\ -(-r + ct) \int_0^\pi \int_0^{2\pi} \alpha(-r + ct, \phi, \theta) \sin \theta \, d\phi \, d\theta & \text{if } r - ct < 0 \end{cases},$$

so for every region in the rt -quarter-plane, we have to test whether $r - ct$ and $r + ct$ are greater than or less than zero.

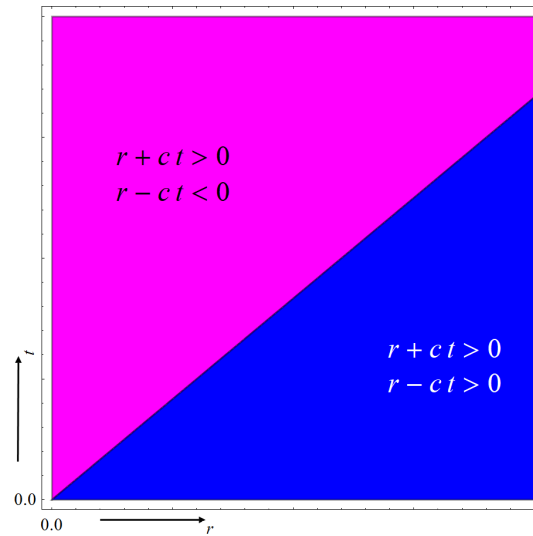


Figure 2: This figure illustrates the regions in the rt -quarter-plane that come about from using the odd extensions of A and B . The solution for w has to be determined in each one. The characteristic curve $r - ct = 0$ is the line that separates the regions.

The Magenta Region

In the magenta region $r + ct > 0$ and $r - ct < 0$, so the solution for w is

$$\begin{aligned}
 w(r, t) &= \frac{1}{2} [A_{\text{odd}}(r + ct) + A_{\text{odd}}(r - ct)] + \frac{1}{2c} \int_{r-ct}^{r+ct} B_{\text{odd}}(s) ds \\
 &= \frac{1}{2} \left[(r + ct) \int_0^\pi \int_0^{2\pi} \alpha(r + ct, \phi, \theta) \sin \theta d\phi d\theta - (-r + ct) \int_0^\pi \int_0^{2\pi} \alpha(-r + ct, \phi, \theta) \sin \theta d\phi d\theta \right] \\
 &\quad + \frac{1}{2c} \left[\int_{r-ct}^0 -(-s) \int_0^\pi \int_0^{2\pi} \beta(-s, \phi, \theta) \sin \theta d\phi d\theta ds + \int_0^{r+ct} s \int_0^\pi \int_0^{2\pi} \beta(s, \phi, \theta) \sin \theta d\phi d\theta ds \right].
 \end{aligned}$$

Substitute $p = -s$ in the third integral and substitute $p = s$ in the fourth integral.

$$\begin{aligned}
 &= \frac{1}{2} \left[\int_0^\pi \int_0^{2\pi} (r + ct) \alpha(r + ct, \phi, \theta) \sin \theta d\phi d\theta - \int_0^\pi \int_0^{2\pi} (-r + ct) \alpha(-r + ct, \phi, \theta) \sin \theta d\phi d\theta \right] \\
 &\quad + \frac{1}{2c} \left[\int_{-r+ct}^0 p \int_0^\pi \int_0^{2\pi} \beta(p, \phi, \theta) \sin \theta d\phi d\theta dp + \int_0^{r+ct} p \int_0^\pi \int_0^{2\pi} \beta(p, \phi, \theta) \sin \theta d\phi d\theta dp \right] \\
 &= \frac{1}{2} \int_0^\pi \int_0^{2\pi} [(r + ct) \alpha(r + ct, \phi, \theta) - (-r + ct) \alpha(-r + ct, \phi, \theta)] \sin \theta d\phi d\theta \\
 &\quad + \frac{1}{2c} \int_{-r+ct}^{r+ct} p \int_0^\pi \int_0^{2\pi} \beta(p, \phi, \theta) \sin \theta d\phi d\theta dp
 \end{aligned}$$

The expression in square brackets can be written as the derivative of an integral by the Leibnitz rule.

$$\begin{aligned}
 &= \frac{1}{2} \int_0^\pi \int_0^{2\pi} \left[\frac{1}{c} \frac{\partial}{\partial t} \int_{-r+ct}^{r+ct} p \alpha(p, \phi, \theta) dp \right] \sin \theta d\phi d\theta + \frac{1}{2c} \int_{-r+ct}^{r+ct} p \int_0^\pi \int_0^{2\pi} \beta(p, \phi, \theta) \sin \theta d\phi d\theta dp \\
 &= \frac{\partial}{\partial t} \left[\frac{1}{2c} \int_{-r+ct}^{r+ct} p \int_0^\pi \int_0^{2\pi} \alpha(p, \phi, \theta) \sin \theta d\phi d\theta dp \right] + \frac{1}{2c} \int_{-r+ct}^{r+ct} p \int_0^\pi \int_0^{2\pi} \beta(p, \phi, \theta) \sin \theta d\phi d\theta dp
 \end{aligned}$$

The Blue Region

In the blue region $r + ct > 0$ and $r - ct > 0$, so the solution for w is

$$\begin{aligned}
 w(r, t) &= \frac{1}{2}[A_{\text{odd}}(r + ct) + A_{\text{odd}}(r - ct)] + \frac{1}{2c} \int_{r-ct}^{r+ct} B_{\text{odd}}(s) ds \\
 &= \frac{1}{2} \left[(r + ct) \int_0^\pi \int_0^{2\pi} \alpha(r + ct, \phi, \theta) \sin \theta d\phi d\theta + (r - ct) \int_0^\pi \int_0^{2\pi} \alpha(r - ct, \phi, \theta) \sin \theta d\phi d\theta \right] \\
 &\quad + \frac{1}{2c} \int_{r-ct}^{r+ct} s \int_0^\pi \int_0^{2\pi} \beta(s, \phi, \theta) \sin \theta d\phi d\theta ds \\
 &= \frac{1}{2} \left[\int_0^\pi \int_0^{2\pi} (r + ct) \alpha(r + ct, \phi, \theta) \sin \theta d\phi d\theta + \int_0^\pi \int_0^{2\pi} (r - ct) \alpha(r - ct, \phi, \theta) \sin \theta d\phi d\theta \right] \\
 &\quad + \frac{1}{2c} \int_{r-ct}^{r+ct} s \int_0^\pi \int_0^{2\pi} \beta(s, \phi, \theta) \sin \theta d\phi d\theta ds \\
 &= \frac{1}{2} \int_0^\pi \int_0^{2\pi} [(r + ct) \alpha(r + ct, \phi, \theta) + (r - ct) \alpha(r - ct, \phi, \theta)] \sin \theta d\phi d\theta \\
 &\quad + \frac{1}{2c} \int_{r-ct}^{r+ct} s \int_0^\pi \int_0^{2\pi} \beta(s, \phi, \theta) \sin \theta d\phi d\theta ds.
 \end{aligned}$$

The expression in square brackets can be written as the derivative of an integral by the Leibnitz rule. Also, let $p = s$ in the second integral.

$$\begin{aligned}
 &= \frac{1}{2} \int_0^\pi \int_0^{2\pi} \left[\frac{1}{c} \frac{\partial}{\partial t} \int_{r-ct}^{r+ct} p \alpha(p, \phi, \theta) dp \right] \sin \theta d\phi d\theta + \frac{1}{2c} \int_{r-ct}^{r+ct} p \int_0^\pi \int_0^{2\pi} \beta(p, \phi, \theta) \sin \theta d\phi d\theta dp \\
 &= \frac{\partial}{\partial t} \left[\frac{1}{2c} \int_{r-ct}^{r+ct} p \int_0^\pi \int_0^{2\pi} \alpha(p, \phi, \theta) \sin \theta d\phi d\theta dp \right] + \frac{1}{2c} \int_{r-ct}^{r+ct} p \int_0^\pi \int_0^{2\pi} \beta(p, \phi, \theta) \sin \theta d\phi d\theta dp
 \end{aligned}$$

Consequently,

$$w(r, t) = \begin{cases} \frac{\partial}{\partial t} \left[\frac{1}{2c} \int_{-r+ct}^{r+ct} p \int_0^\pi \int_0^{2\pi} \alpha(p, \phi, \theta) \sin \theta \, d\phi \, d\theta \, dp \right] + \frac{1}{2c} \int_{-r+ct}^{r+ct} p \int_0^\pi \int_0^{2\pi} \beta(p, \phi, \theta) \sin \theta \, d\phi \, d\theta \, dp & \text{if } r - ct < 0 \\ \frac{\partial}{\partial t} \left[\frac{1}{2c} \int_{r-ct}^{r+ct} p \int_0^\pi \int_0^{2\pi} \alpha(p, \phi, \theta) \sin \theta \, d\phi \, d\theta \, dp \right] + \frac{1}{2c} \int_{r-ct}^{r+ct} p \int_0^\pi \int_0^{2\pi} \beta(p, \phi, \theta) \sin \theta \, d\phi \, d\theta \, dp & \text{if } r - ct > 0 \end{cases}.$$

Since $v = w/r$, we have

$$\int_0^\pi \int_0^{2\pi} u(r, \phi, \theta, t) \sin \theta \, d\phi \, d\theta = \frac{w(r, t)}{r}.$$

In order to calculate the value of u at the origin, take the limit of both sides as $r \rightarrow 0$.

$$\lim_{r \rightarrow 0} \int_0^\pi \int_0^{2\pi} u(r, \phi, \theta, t) \sin \theta \, d\phi \, d\theta = \lim_{r \rightarrow 0} \frac{w(r, t)}{r}$$

The value of u at $r = 0$ is $u(x = 0, y = 0, z = 0, t)$. It does not depend on ϕ and θ , so it can be pulled in front of the integral.

$$u(0, 0, 0, t) \int_0^\pi \int_0^{2\pi} \sin \theta \, d\phi \, d\theta = \lim_{r \rightarrow 0} \frac{w(r, t) - w(0, t)}{r - 0}$$

Evaluate the integral on the left side. The limit on the right side is how the first derivative of w with respect to r at $r = 0$ is defined. The formula for w in the magenta region applies for this value of r . Use the Leibnitz rule to differentiate the integrals with respect to r .

$$\begin{aligned} 4\pi u(0, 0, 0, t) &= \left. \frac{\partial w}{\partial r} \right|_{r=0} \\ &= \left. \left\{ \frac{\partial}{\partial t} \left[\frac{1}{2c} \frac{\partial}{\partial r} \int_{-r+ct}^{r+ct} p \int_0^\pi \int_0^{2\pi} \alpha(p, \phi, \theta) \sin \theta \, d\phi \, d\theta \, dp \right] + \frac{1}{2c} \frac{\partial}{\partial r} \int_{-r+ct}^{r+ct} p \int_0^\pi \int_0^{2\pi} \beta(p, \phi, \theta) \sin \theta \, d\phi \, d\theta \, dp \right\} \right|_{r=0} \\ &= \left. \left\{ \frac{1}{2c} \frac{\partial}{\partial t} \left[(r + ct) \int_0^\pi \int_0^{2\pi} \alpha(r + ct, \phi, \theta) \sin \theta \, d\phi \, d\theta - (-r + ct)(-1) \int_0^\pi \int_0^{2\pi} \alpha(-r + ct, \phi, \theta) \sin \theta \, d\phi \, d\theta \right] \right. \right. \\ &\quad \left. \left. + \frac{1}{2c} \left[(r + ct) \int_0^\pi \int_0^{2\pi} \beta(r + ct, \phi, \theta) \sin \theta \, d\phi \, d\theta - (-r + ct)(-1) \int_0^\pi \int_0^{2\pi} \beta(-r + ct, \phi, \theta) \sin \theta \, d\phi \, d\theta \right] \right\} \right|_{r=0} \\ &= \frac{1}{2c} \frac{\partial}{\partial t} \left[ct \int_0^\pi \int_0^{2\pi} \alpha(ct, \phi, \theta) \sin \theta \, d\phi \, d\theta + ct \int_0^\pi \int_0^{2\pi} \alpha(ct, \phi, \theta) \sin \theta \, d\phi \, d\theta \right] \\ &\quad + \frac{1}{2c} \left[ct \int_0^\pi \int_0^{2\pi} \beta(ct, \phi, \theta) \sin \theta \, d\phi \, d\theta + ct \int_0^\pi \int_0^{2\pi} \beta(ct, \phi, \theta) \sin \theta \, d\phi \, d\theta \right] \end{aligned}$$

As a result,

$$\begin{aligned} 4\pi u(0, 0, 0, t) &= \frac{\partial}{\partial t} \left[t \int_0^\pi \int_0^{2\pi} \alpha(ct, \phi, \theta) \sin \theta \, d\phi \, d\theta \right] + t \int_0^\pi \int_0^{2\pi} \beta(ct, \phi, \theta) \sin \theta \, d\phi \, d\theta \\ &= \frac{\partial}{\partial t} \left[\frac{1}{c^2 t} \int_0^\pi \int_0^{2\pi} \alpha(ct, \phi, \theta) (c^2 t^2 \sin \theta \, d\phi \, d\theta) \right] + \frac{1}{c^2 t} \int_0^\pi \int_0^{2\pi} \beta(ct, \phi, \theta) (c^2 t^2 \sin \theta \, d\phi \, d\theta). \end{aligned}$$

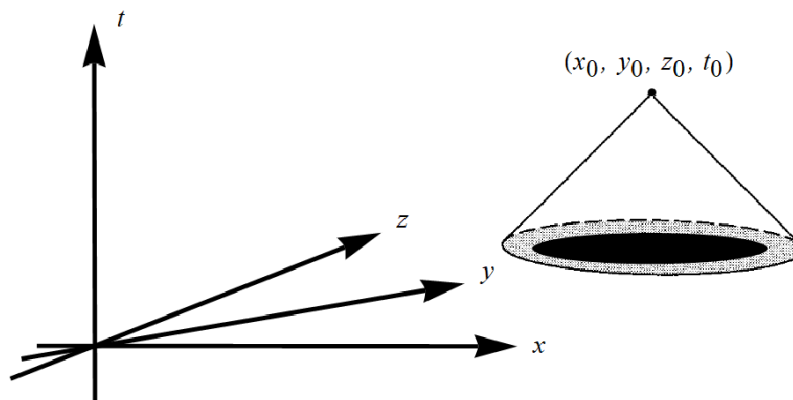
Divide both sides by 4π .

$$u(0, 0, 0, t) = \frac{\partial}{\partial t} \left[\frac{1}{4\pi c^2 t} \int_0^\pi \int_0^{2\pi} \alpha(ct, \phi, \theta) (c^2 t^2 \sin \theta \, d\phi \, d\theta) \right] + \frac{1}{4\pi c^2 t} \int_0^\pi \int_0^{2\pi} \beta(ct, \phi, \theta) (c^2 t^2 \sin \theta \, d\phi \, d\theta)$$

At a particular time $t = t_0$ these double integrals are surface integrals over a sphere of radius ct_0 centered at $(0, 0, 0)$.

$$u(0, 0, 0, t_0) = \frac{\partial}{\partial t_0} \left[\frac{1}{4\pi c^2 t_0} \iint_{x^2+y^2+z^2=c^2 t_0^2} \alpha(x, y, z) \, dS \right] + \frac{1}{4\pi c^2 t_0} \iint_{x^2+y^2+z^2=c^2 t_0^2} \beta(x, y, z) \, dS$$

This is the solution of the wave equation at the origin of the xyz -plane. Now we aim to find the solution at a particular point in space-time (x_0, y_0, z_0, t_0) .



The wave equation is invariant to translations in space, so if $u(x, y, z, t)$ is a solution to the wave equation

$$\begin{aligned} u_{tt} &= c^2 \nabla^2 u, \quad -\infty < x, y, z < \infty, \quad t > 0 \\ u(x, y, z, 0) &= \alpha(x, y, z) \\ u_t(x, y, z, 0) &= \beta(x, y, z), \end{aligned}$$

then $u(x + x_0, y + y_0, z + z_0, t)$ is also a solution to the wave equation, albeit with different initial conditions.

$$\begin{aligned} u_{tt} &= c^2 \nabla^2 u, \quad -\infty < x, y, z < \infty, \quad t > 0 \\ u(x + x_0, y + y_0, z + z_0, 0) &= \alpha(x + x_0, y + y_0, z + z_0) \\ u_t(x + x_0, y + y_0, z + z_0, 0) &= \beta(x + x_0, y + y_0, z + z_0) \end{aligned}$$

Since the solution to the wave equation is unique, $u(x_0, y_0, z_0, t_0)$ has the same form as $u(0, 0, 0, t_0)$.

$$u(x_0, y_0, z_0, t_0) = \frac{\partial}{\partial t_0} \left[\frac{1}{4\pi c^2 t_0} \iint_{x^2+y^2+z^2=c^2 t_0^2} \alpha(x+x_0, y+y_0, z+z_0) dS \right] + \frac{1}{4\pi c^2 t_0} \iint_{x^2+y^2+z^2=c^2 t_0^2} \beta(x+x_0, y+y_0, z+z_0) dS$$

Let $k = x + x_0$, $l = y + y_0$, and $m = z + z_0$. Then $x = k - x_0$, $y = l - y_0$, and $z = m - z_0$.

$$u(x_0, y_0, z_0, t_0) = \frac{\partial}{\partial t_0} \left[\frac{1}{4\pi c^2 t_0} \iint_{\substack{(k-x_0)^2+(l-y_0)^2 \\ +(m-z_0)^2=c^2 t_0^2}} \alpha(k, l, m) dS \right] + \frac{1}{4\pi c^2 t_0} \iint_{\substack{(k-x_0)^2+(l-y_0)^2 \\ +(m-z_0)^2=c^2 t_0^2}} \beta(k, l, m) dS$$

k , l , and m are dummy integration variables, so they can be replaced with x , y , and z .

$$u(x_0, y_0, z_0, t_0) = \frac{\partial}{\partial t_0} \left[\frac{1}{4\pi c^2 t_0} \iint_{\substack{(x-x_0)^2+(y-y_0)^2 \\ +(z-z_0)^2=c^2 t_0^2}} \alpha(x, y, z) dS \right] + \frac{1}{4\pi c^2 t_0} \iint_{\substack{(x-x_0)^2+(y-y_0)^2 \\ +(z-z_0)^2=c^2 t_0^2}} \beta(x, y, z) dS$$

Finally, switch the roles of x , y , z , and t with those of x_0 , y_0 , z_0 , and t_0 , respectively, to obtain the legendary solution to the initial value problem (discovered by Kirchhoff & Poisson).

$$u(x, y, z, t) = \frac{\partial}{\partial t} \left[\frac{1}{4\pi c^2 t} \iint_{\substack{(x_0-x)^2+(y_0-y)^2 \\ +(z_0-z)^2=c^2 t^2}} \alpha(x_0, y_0, z_0) dS_0 \right] + \frac{1}{4\pi c^2 t} \iint_{\substack{(x_0-x)^2+(y_0-y)^2 \\ +(z_0-z)^2=c^2 t^2}} \beta(x_0, y_0, z_0) dS_0$$

These double integrals are surface integrals over a sphere of radius ct centered at (x, y, z) . In this exercise $\alpha = 0$ and $\beta = y$, so the formula simplifies to

$$u(x, y, z, t) = \frac{1}{4\pi c^2 t} \iint_{\substack{(x_0-x)^2+(y_0-y)^2 \\ +(z_0-z)^2=c^2 t^2}} y_0 dS_0.$$

Use spherical coordinates (r_0, ϕ_0, θ_0) to evaluate the integral.

$$\begin{aligned} x_0 - x &= ct \sin \theta_0 \cos \phi_0 \\ y_0 - y &= ct \sin \theta_0 \sin \phi_0 \quad \rightarrow \quad y_0 = y + ct \sin \theta_0 \sin \phi_0 \\ z_0 - z &= ct \cos \theta_0 \end{aligned}$$

As a result,

$$\begin{aligned}
 u(x, y, z, t) &= \frac{1}{4\pi c^2 t} \int_0^\pi \int_0^{2\pi} (y + ct \sin \theta_0 \sin \phi_0) (c^2 t^2 \sin \theta_0 \, d\phi_0 \, d\theta_0) \\
 &= \frac{t}{4\pi} \int_0^\pi \int_0^{2\pi} (y \sin \theta_0 + ct \sin^2 \theta_0 \sin \phi_0) \, d\phi_0 \, d\theta_0 \\
 &= \frac{t}{4\pi} \left(y \int_0^\pi \int_0^{2\pi} \sin \theta_0 \, d\phi_0 \, d\theta_0 + ct \int_0^\pi \int_0^{2\pi} \sin^2 \theta_0 \sin \phi_0 \, d\phi_0 \, d\theta_0 \right) \\
 &= \frac{t}{4\pi} \left[y \left(\int_0^\pi \sin \theta_0 \, d\theta_0 \right) \left(\int_0^{2\pi} d\phi_0 \right) + ct \left(\int_0^\pi \sin^2 \theta_0 \, d\theta_0 \right) \underbrace{\left(\int_0^{2\pi} \sin \phi_0 \, d\phi_0 \right)}_{=0} \right] \\
 &= \frac{t}{4\pi} [y(2)(2\pi)].
 \end{aligned}$$

Therefore, the solution to the initial value problem with $\alpha = 0$ and $\beta = y$ is

$$u(x, y, z, t) = ty.$$